The **ScienceMaths** project: **Logarithmic function**
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Teaching Material

including

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The loudness of sound

Sound intensity is the amount of energy which moves past a given area of the medium per unit of time. The energy/time ratio is equivalent to the quantity power. Sound power is the energy rate - the energy of the sound per unit of time from the source of the sound. Therefore sound intensity is defined as the power per unit area.

Power

$$Intensity = \frac{Energy}{Time * Area} \qquad Intensity = \frac{Power}{Area}$$

$$I = \frac{W}{t * S} \qquad I = \frac{P}{S}$$

Amount	Symbol	Unit	Name of units
Energy	W	J	Joule
Area	S	m^2	Square meter
Time	T	S	Second
Power	P	J/s = W	Watt
Intensity	I	J/s m^2 = W/m^2	Watt/square meter

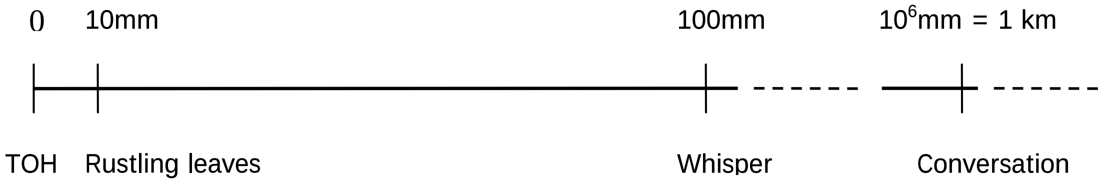
Table 1: Symbols and units of certain amounts.

Human are equipped with very sensitive ears, capable of detecting sound waves of extremely low intensity of only 10^{-12} W/m^2. This is known as the threshold of hearing (TOH). The highest sound intensity which the ear can safely detect without pain, suffering, and physical damage is 1 W/m^2. This means 1 000 000 000 000 (10^{12}) times as loud as the quietest. This range makes the absolute value of the sound intensity impractical for normal use.

Task: Imagine that the speedometer in a car can detect such a big range of speed (10^{12}). If the maximum velocity shown is 200 km per hour, what is the minimum velocity?

Answer: 200 km/h·10^{-12} = 0.2·10^{-3} mm/h = 4.8·10^{-3} mm/day = 1.752 mm/year
This means that the scale should show a velocity from 0.0000000002 km/hour (1.8 mm/year) to 200 km/hour (1 800 000 000 000 mm/year = 1.8·10^{12} mm/year).

Let us return to sound. As we have already said, the lowest sound that can be heard by the human ear is a sound that corresponds to the intensity 10^{-12} W/m^2. Rustling leaves are 10 times and a whisper is 100 times more intense (10^{-10} W/m^2), the sound of normal conversation is a million times more intense: 10^{-6} W/m^2. If we try to present this data on a normal scale, then it looks like this:



Normal conversation should be drawn on a scale 1 000 000 mm away from the TOH – it is 1 km away and a walkman at maximum level, which is 10^{10} times more intense than the quietest sound, is 10 000 000 000 mm away – i.e., 10 000 km away –approximately 1/4 of the circumference of the earth (the distance between London and Singapore or London and Los Angeles).
The threshold of pain is 10 W/m^2. It is 10^{13} times more intense than the TOH. We should draw it 10 000 000 km away (The distance from the Earth to the Moon is 286 000 km.)
Since the range of intensities which the human ear can detect is so large, it is difficult to present it by such a scale. Therefore we introduce the logarithm of the ratio of the intensity of sound (I) to the threshold intensity (I_o):

$$\log \frac{I}{I_0}$$

The intensity level is expressed in units named Bels[1]. Since rustling leaves are 10 times more intense than the TOH, the intensity level of rustling leaves is:

$$\log \frac{10 * I_0}{I_0} = \log 10 = 1 Bel$$

Since a whisper is 100 times more intense than TOH, the intensity level of a whisper is:

$$\log \frac{100 * I_0}{I_0} = \log 100 = \log 10^2 = 2 Bels$$

Task: How many Bels is normal conversation?

Answer: Normal conversation is a million times more intense than the TOH. Its intensity level is:

$$\log \frac{1000000 * I_0}{I_0} = \log 1000000 = \log 10^6 = 6 Bels$$

The logarithm involved is just the power of ten of the sound intensity expressed as a multiple of the threshold of hearing intensity. When a sound is 10 times as intense as another one, its loudness is 1 Bel greater. If a sound is 100 times as intense, it is louder by 2 Bels, and so on.

A Bel is a large unit, so a subunit, a *decibel,* is usually used. About 0.1 Bels is the just noticeable difference in sound intensity for the normal human ear. 0.1 Bels is equal to 1

[1] Bels are named after Alexander Graham Bell (1847 – 1922) - scientist, inventor, and innovator. Bell is widely acclaimed for inventing and developing the telephone in 1876.

decibel (dB). This makes the decibel a conveniently sized unit and the factor of 10 multiplying the logarithm in the formula makes it decibels instead of Bels:

$$I(dB) = 10\log_{10}\left(\frac{I}{I_0}\right)$$

In fact, the purpose of using the factor 10 in the definition of the decibel is to create a unit which is approximately the smallest detectable change in sound intensity. The human ear interprets changes in loudness within a logarithmic scale. The sensation of loudness of sound is not proportional to the energy intensity, but is rather a logarithmic function.

The intensity level of the threshold of hearing:

$$I(dB) = 10\log_{10}\left(\frac{I_0}{I_0}\right) = 10\log_{10} 1 = 0dB$$

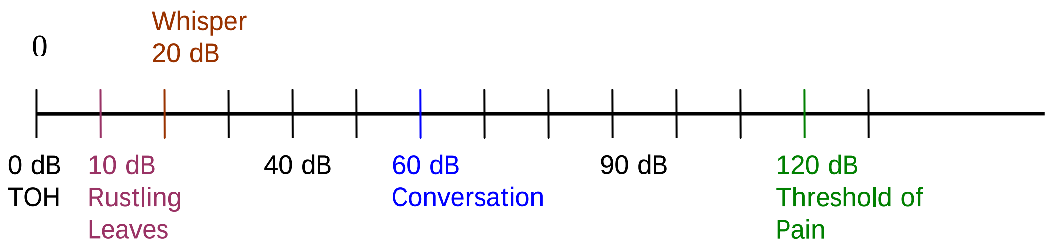
The intensity level of rustling leaves:

$$I(dB) = 10\log_{10}\left(\frac{10I_0}{I_0}\right) = 10\log_{10} 10 = 10dB$$

The intensity level of normal conversation:

$$I(dB) = 10\log_{10}\left(\frac{1000000I_0}{I_0}\right) = 10\log_{10} 10^6 = 60dB$$

Now instead of the classical scale (as we saw before), we can draw a logarithmic scale:



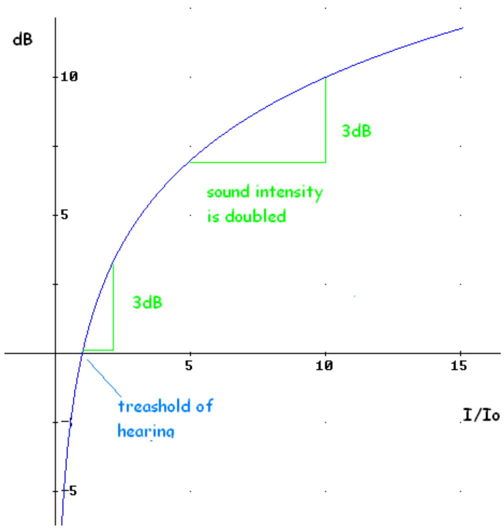
If we increase the sound intensity by a factor 10, its level increases by 10 dB:

$$I(dB) = 10\log_{10}\left(\frac{10I_0}{I_0}\right) = 10\log_{10} 10 = 10dB$$

Task: How much does the level of sound intensity increase if we double the sound intensity ($I = 2\ I_0$)?

Answer: If we double the sound intensity ($I = 2\ I_0$), its level increases by 3 dB.

$$I(dB) = 10\log_{10}\left(\frac{2I_0}{I_0}\right) = 10\log_{10} 2 = 3dB$$



Task: Take a sound level meter and measure the intensity of your talking. Doubled by your feeling the intensity of your conversation and measure the intensity again.

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Map of the Universe

A logarithmic scale is also very useful for presenting the enormous distances in our universe. We will try to draw a map of the universe.

Let us first look at the distances between the Earth and some other places in the universe.

The distance from the Earth to:

	Light second	Light minute	Light year
Moon (286 000 km)	0.95	0.016	0.0000000302
Sun (149 000 000 km)	496.67	8.278	0.0000160000
Saturn	1860.00	31.000	0.0000059000
Proxima Centauri[2]	135604800.00	2260080.000	4.3000000000
Center of Milky Way	819936000000.00	13 665 600 000.000	26 000.0000000000
Andromeda[3]	69379200000000.00	1156320000 000.000	2 200 000.0000000000

Table 2: Distances from the Earth to different objects in the universe

Task: Take one roll of toilet paper. The length of one sheet would represent the distance from the Earth to the Moon. Unroll the roll and mark where the Sun is! (Check how many sheets are in one roll!)

Answer: The distance to the Sun is more than 500 times the distance to the Moon. Thus if you were to unroll the entire roll of bathroom tissue, the length of one sheet would represent the distance from the Earth to the Moon, and the whole roll would not be enough for the distance from the Earth to the Sun (if one roll contains 400 sheets). Along the way, we would pass Venus around sheet 135 and Mercury at about sheet 260.

If we try to make a map and draw the Moon 1 cm away from the Earth, then the Sun should be drawn 5 m away, Saturn almost 20 m, Proxima Centauri 1422 km away, and the center of the Milky Way Galaxy 8 600 727 km away – our Moon (not on the map) is only 286 000 km away.

The Moon	1.0 cm	1 cm
The Sun	521.0 cm	5.2 m
Sarturn	1951.0 cm	19.5 m
Proxima Centaruri	142242797.2 cm	1422.4 km
The center of the Milky Way	8.6 E+11 cm	8 600 727.2 km
Andromeda	7.3 E+13 cm	727 753 846.2 km

Table 3: Distances between the above objects and Earth on a map – at scale of 1 to 28 600 000 000[4].

[2] The nearest star to the Sun.
[3] The only other galaxy which can be seen with the naked eye.
[4] 1 cm on the map is equal 28 600 000 000 cm in nature.

So, let us draw the Moon only 1 mm away from the Earth. The Sun is then half a meter away, Saturn 1.9 m, but the center of our galaxy is 860 073 km away.

That means that we face the same problem. We can not draw such distances in any manner if we use normal proportions. There would be no end to such an attempt. And we can not draw the map of the universe, or even a small part of it.

So the only way to succeed is to use a logarithmic scale again.

Task: Try to draw a map of the universe by using a logarithmic scale!

Answer: The Moon is 1 cm away, the Sun 2.7 cm, Saturn 3.3 cm, Proxima Centauri 8.2 cm, the center of the Milky Way 11.9 cm, and Andromeda 13.9 cm.

Earthquakes

An earthquake is a movement of the Earth's surface. It occurs where the tectonic plates forming the earth's surface meet. The plates move past, towards, or away from each other, but friction can cause them to get stuck, resulting in a build-up of pressure. When the pressure is released, an earthquake occurs.
As the pressure is released it produces shock waves called seismic waves.
Magnitude is the word used to describe the strength of an earthquake.

The magnitude of most earthquakes is measured on the Richter scale, invented by Charles F. Richter in 1934. A Richter magnitude is calculated from the amplitude of the largest seismic wave recorded for an earthquake.

Richter magnitudes[5] are based on a logarithmic scale. What this means is that for each increasing whole number on the Richter scale, the amplitude of the ground motion recorded by a seismograph goes up ten times. This means that an earthquake of magnitude 5 is 10 times stronger than an earthquake of magnitude 4, and one of magnitude 6 is 100 times stronger than an earthquake of magnitude 4.

The Richter magnitude scale can be used to describe earthquakes so small that they are expressed in negative numbers. The scale also has no upper limit, so it can describe earthquakes of unimaginable and (so far) never experienced intensity, such as magnitude 10 and beyond.

Richter Magnitudes	Number of such earthquakes per year	Earthquake Effects
< 3.5	800 000	Generally not felt, but recorded (detected by seismographs)
3.5 – 4.2	30 000	Just barely noticeable indoors
4.3 – 4.8	4 800	Most people notice them, windows rattle
4.9 – 5.4	1 400	Often felt (open doors swing), but rarely causes damage
5.5 – 6.1	500	Slight damage, to buildings, plaster cracks, bricks fall
6.2 – 6.9	100	Much damage to buildings, chimneys fall, houses move on foundations, can be destructive in areas up to about 100 kilometers in diameter where people live.
7.0 – 7.3	15	Serious damage over larger areas, bridges twist, walls fracture, building may collapse
7.4 – 7.9	4	Great damage, most buildings collapse
> 8.0	One every 5 to 10 years	Total damage, surface waves seen, objects thrown in the air

Table 4: How human beings feel the intensity of an earthquake.

5 $M = \log\left(\frac{A}{A_0}\right)$, where A is the biggest amplitude, A_0 is the normalized factor.

Earthquakes cause many deaths:

Year	Place	Magnitude	Deaths
2006	Indonesia	6.3	5 749
2005	Pakistan	7.6	86 000
2005	Northern Sumatra, Indonesia	8.6	1 313
2004	Sumatra	9.1	283 106
2003	Southeastern Iran	6.6	31 000
2003	Northern Algeria	6.8	2 266
2002	Hindu Kush Region, Afghanistan	6.1	1 000
2001	Gujarat, India	7.6	20 085
1999	Taiwan	7.6	2 400
1999	Turkey	7.6	17 118
1999	Colombia	6.1	1 185
1998	Papua New Guinea	7.0	2 183
1998	Afghanistan-Tajikistan Border Region	6.6	4 000
1998	Hindu Kush region, Afghanistan	5.9	2 323
1997	Northern Iran	7.3	1 567

Table 4: Earthquake with 1000 or more deaths in the last 10 years.

Task: Find all significant earthquakes (Magnitude > 5) in the world this year!

Answer: http://earthquake.usgs.gov/eqcenter/eqarchives/significant/

Task: Find strong earthquakes in your country!

Answer: Strong earthquakes in Slovenia: 1348 Beljak, (damaged 26 towns, 40 castles and churches, 20000 deaths), 1511 Idrija, (damaged many castles, 12,000 deaths), 1690 Beljak, 1889 Zagrebska gora, 1895 Ljubljana, 1917 Brezice, 1956 Ilirska Bistrica, 1963 Litija, 1974 Kozjansko, 1976 Furlanija, Posocje (damaged 12,000 buildings), 1995 Ilirska Bistrica , 1998 Posocje (5.6), 2004 Posocje (4.2).

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Task: The magnitude of the earthquake in Slovenia (Posocje) on Easter Sunday 1998 (April 12th) was 5.6. Fortunately no one died. Ten days earlier the same year there was an earthquake in Afghanistan of magnitude 5.9. In that quake 2323 people died. How much stronger was the earthquake in Afghanistan than the one in Slovenia?

Answer: Slovenia: $\log_{10} s = 5.6$ $\quad s = 10^{5.6} = 398\,107$

Afghanistan: $\log_{10} a = 5.9$ $\quad a = 10^{5.9} = 794\,328$

The earthquake in Afghanistan is almost two times as strong as the earthquake in Slovenia because:

$$\frac{794328}{398107} = 1.995$$

Task: How much stronger was the earthquake in Slovenia in 1998 (5.6) than in 2004 (4.2)?

Answer: More than 25 times, because:

$$\frac{10^{5.6}}{10^{4.2}} = \frac{3.981071705 \cdot 10^{5}}{1.584893192 \cdot 10^{4}} = 25.12$$

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pH measurement

Acids produce hydrogen ions (H^+). Acids such as hydrochloric acid produce many hydrogen ions; this is because when hydrogen chloride gas dissolves in water, the hydrogen chloride molecules break up into hydrogen ions and chloride ions.

$$HCl = H^+ + Cl^-$$

Water also breaks up to produce ions, hydrogen ions and hydroxyl ions.

$$H_2O = H^+ + OH^-$$

In each case we can measure or calculate the concentration of hydrogen ions present. We indicate this with the symbol $[H^+]$; we use square brackets to show that it is the concentration of hydrogen ions. $[H^+]$ is the concentration of H^+ ions in moles per liter (a mole is a unit of measurement, equal to 6.022×10^{23} atoms).
Tremendous swings in the hydrogen ion (hydronium ion) concentration occur in water when acids or bases are mixed with water. These changes can be as big as 1×10^{14}, This means that concentrations can change by multiples as big as one hundred trillion, 100,000,000,000,000.

Big changes once again. That is why we introduce a new scale, a logarithm scale.
We define[6]:

$$pH = -\log[H^+]$$

where $[H^+]$ is the molar concentration of hydronium ions, M = moles / liter.

Because H^+ ions associate with water molecules to form hydronium (H_3O^+) ions, pH is often expressed in terms of the concentration of hydronium ions:

$$pH = -\log[H_3O^+]$$

In water at 22°C, H_3O^+ and hydroxyl (OH^-) ions exist in equal quantities; the concentration of each is 1.0×10^{-7} moles per liter (mol/L). Therefore the pH of water is equal to:

$$pH = -\log 10^{-7} = 7$$

In an HCl (hydrochloric acid) solution, $[H^+] = 0.01 = 10^{-2}$, therefore

$$pH = -\log 10^{-2} = 2$$

If the pH is low, it means that there is a high concentration of hydrogen ions and if the pH is high it means that there is a very low concentration of hydrogen ions or none at all.

As was already stated, water breaks up to produce ions, hydrogen ions and hydroxyl ions.

$$H_2O = H^+ + OH^-$$

The molar concentration of both hydronium ions and hydroxyl ions is the same:

[6] p is an operator which is defined as the log of whatever quantity follows the symbol. It communicates the instruction to calculate the negative log of any quantity that follows the symbol.

$$[H^+] = 0.0000001 = 10^{-7} = [OH^-]$$

The product is:

$$[H^+]\,[OH^-] = 10^{-7} \cdot 10^{-7} = 10^{-14}$$

If we use the logarithm function on both sides:

$$-\log([H^+]\,[OH^-]) = -\log 10^{-14}$$

We know that the log of the product is equal to the sum of the log ($\log_a(xy) = \log_a x + \log_a y$):

$$-\log [H^+] - \log [OH^-] = -\log 10^{-14}$$

We get the relationship:

$$pH + pOH = 14$$

Sodium hydroxide also breaks up to produce ions when it is dissolved in water, sodium ions and hydroxyl ions.

$$NaOH = Na^+ + OH^-$$

In a NaOH (Sodium Hydroxide) solution

$[H^+] = 0.00000000000001 = 10^{-14}$ therefore $pH = -\log 10^{-14} = 14$

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Concentration of hydrogen ions $[H^+]$	Concentration of hydroxyl ions $[OH^-]$	Concentration of hydrogen ions compared to distilled water	Solutions	pH
10^0	10^{-14}	10 000 000	Battery acid	**0**
10^{-1}	10^{-13}	1 000 000	Hydrochloric acid	**1**
10^{-2}	10^{-12}	100 000	Lemon juice, vinegar	**2**
10^{-3}	10^{-11}	10 000	Orange juice, soda, cola	**3**
10^{-4}	10^{-10}	1 000	Tomato juice, acid rain	**4**
10^{-5}	10^{-9}	100	Soft drinking water, coffee	**5**
10^{-6}	10^{-8}	10	Urine	**6**
10^{-7}	10^{-7}	1	Pure water	**7**
10^{-8}	10^{-8}	1/10	Sea water, egg whites	**8**
10^{-9}	10^{-5}	1/100	Baking soda	**9**
10^{-10}	10^{-4}	1/1 000	The Great Salt Lake	**10**
10^{-11}	10^{-3}	1/10 000	Ammonia solution	**11**
10^{-12}	10^{-2}	1/100 000	Soapy water	**12**
10^{-13}	10^{-1}	1/1 000 000	Bleaches, oven cleaner	**13**
10^{-14}	10^0	1/10 000 000	Liquid drain cleaner	**14**

Table 5: pH of different solutions.

Task: What is the pH of a solution whose $[H_3O^+] = 2.5 \times 10^{-5}$ M?

Answer: $pH = -\log[H_3O^+] = -\log[2.5 \times 10^{-5}] = -[\log 2.5 + \log 10^{-5}]$
$\log 2.5$ calculate with a calculator, $\log 10^{-5} = -5$:
$pH = -[0.3979 - 5] = 4.6021$

Task: Calculate the pH of a solution that has a $[OH^-] = 1 \times 10^{-5}$ M

Answer: Use the relationship pH + pOH = 14, therefore pH = 9.

The gentleman who invented the pH scale wanted to measure how acidic his beer was. As we know, yeast is used to make beer or wine, and yeast uses enzymes. Enzymes only work if the pH is right. What he discovered was that acids and alkalis make the colors in plants change. Try grinding up some red cabbage or a beetroot. You will get a red or blue liquid. You can make the liquid change color by adding acids or alkalis. All that happens is that the color of the plant dye depends upon the concentration of Hydrogen ions. If you have used universal indicator paper (the most common one is litmus paper), you know that it can change from blue to green, yellow, or red. A universal indicator is a mixture of dyes which all change when you change the pH.
Since pH is a base 10 log scale, the pH changes by 1 for every power of 10 change in $[H^+]$. A solution of pH 3 has an H^+ concentration 10 times that of a solution of pH 4. The pH

scale ranges from 0 to 14. However, pH values less than 0 and greater than 14 have been observed in very rare concentrated solutions.

Task: Take 1 dl of lemon juice. Measure the pH of it. It should be 2. Try to add so much pure water that you change the pH from 2 to 3. Taste it! How acidic is it?

Answer: You need 9 dl of pure water. The new solution is as acidic as orange juice. Taste both.

Task: Imagine that you have had an accident. You poured one ml of battery acid (pH 0) on your hand. How much water do you need to pour on your hand so that you change the pH from 0 to 4?

Answer: You need 9 ml to change the pH from 0 to 1, 99 ml from 1 to 2, 999 ml from 2 to 3, 9999 ml from 3 to 4, and 99999 ml (= 99.999) l from 4 to 5.