The **ScienceMath**-project: **Growth**
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Growth

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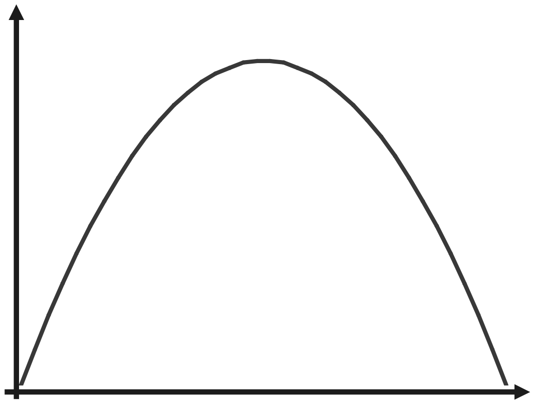
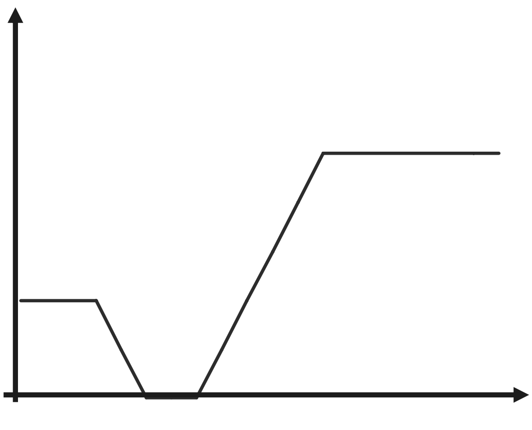
1 Graphs

Graphs are often used in order to describe situations from everyday life, and in this section we look at some examples.

1.1: Try to imagine a situation from everyday life which the graph below describes. Explain with your own words what goes on in this situation. Remember to state which quantities assign to the two axes.



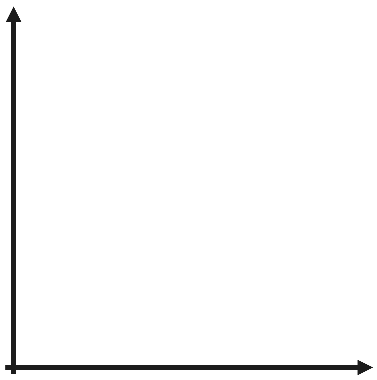
1.2: Try to imagine a situation from everyday life which the graph below describes. Explain with your own words what goes on in this situation. Remember to state which quantities assign to the two axes.



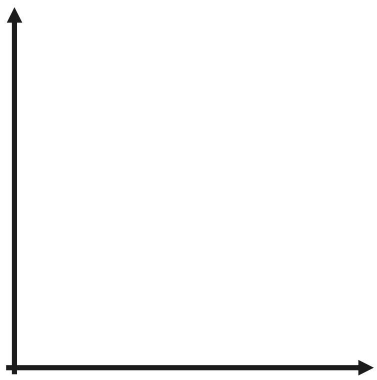
1.3: In each of the following cases you must sketch a graph which can represent the situation described. Before you draw the graph you should carefully consider which quantities you will allot to the axes.

(a) You open the hot water faucet. The temperature of the running water depends on the amount of time passed since you opened the faucet.

(b) You drop a plastic ball out of a window in a 2nd floor flat. The height of the ball from the street level depends on the time passed since you dropped the ball.



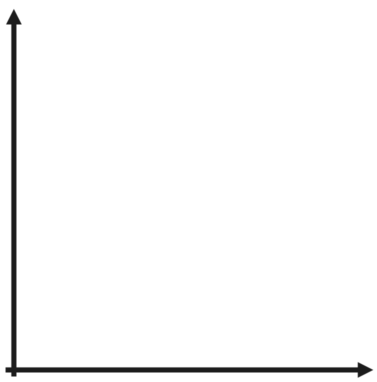
(a)



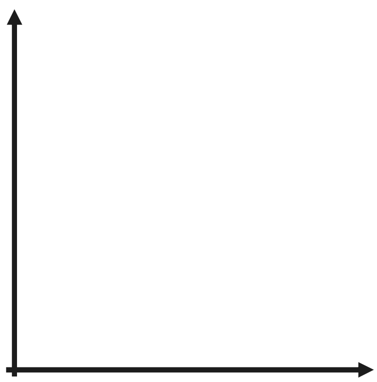
(b)

(c) You are in bright sunlight and walk into a dark room. The diameter of your pupils depends on how long you've been in the dark room.

(d) You deposit a large amount of money on a bank account with a fixed interest. The amount of money on that account depends on how long ago you deposited the money.



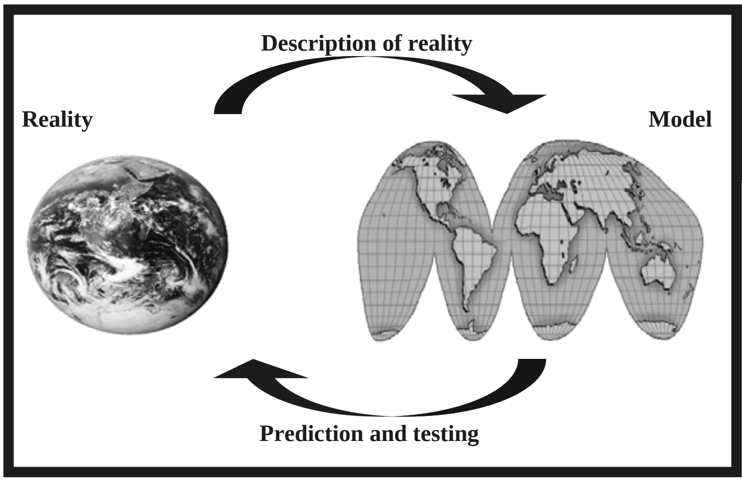
(c)



(d)

2 Mathematical models

An important reason to learn mathematics is that in doing so one acquires methods to solve real-life problems. The problem areas that are described using mathematics are often immensely complex. Therefore it can be necessary to simplify and idealize the situation. This is why mathematical descriptions of real-life situations are called *mathematical models* of reality



There are other models than mathematical models. A world map is also a model of reality.

The construction and application of a mathematical model is usually a process in which the individual steps must be repeated. The first model typically yields some predictions on the problem are at hand. These predictions can be tested by the collection of data. This test may, in turn, lead to improvements on the model and thereby to new predictions, which, again, can be tested. Repeating this process often leads to very precise predictions on real-life situations.

The construction of a mathematical model often includes one of the following description methods.

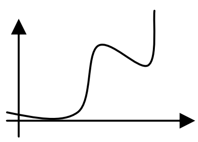
1. One can give a *numerical* description. Here one would usually arrange a series of data in a table which describes a specific development.
2. One can give a *symbolic* description. Here one would use mathematical symbols and expressions to describe a specific development
3. One can give a *graphical* description. Here one would describe a situation by means of a graph in a system of coordinates.

Height	160	171	172
Weight	66	68	75

Numerical method

$$P = \alpha \cdot t + \beta$$

Symbolic method



Graphical method

Verbal Models

Before the construction of a mathematical model of a situation it is a good idea to describe that situation in words. In that way it can be easier to see how the mathematical model can be constructed. This is why some mathematical models begin as *rules of thumb* – a simple description with words of a situation in real-life. A good example of a rule of thumb – or a verbal model – is:

> "Your height at age 2 is half your eventual height as an adult"

Having constructed such a verbal model it is possible to continue to construct a mathematical model on the basis of the verbal model.

2.1: Give examples of mathematical models. State where you know these models from and what use they have. Use verbal models, and numeric, graphical and symbolic methods to illuminate the models.

Here is a rule of thumb:

> ”The deeper a diver dives, the greater atmospheric pressure is she exposed to.
> When the depth is increases by 10 metres, the pressure increases by 1 atm”

2.2: Suppose that the pressure at sea level is 1 atm. Give a mathematical description of the rule of thumb mentioned above.

Mathematical models in which the relationship between two quantities can be described by a straight line in a system of coordinates are called *linear models*.

2.3: Which methods can be used to determine whether the relationship between two quantities can be described with a linear model?

2.4: Give examples of linear models. State where you know the models from, and what they can be used for.

The table below is found in the English mathematics book *Mathematical Modeling* by J. Berry and K. Houston:

Example 2 World Record for the Mile

Table 1.2 shows the world record for the mile in minutes and seconds between 1913 and 1986

Time	Name	Country	Date	Place
4:14.4	John Paul Jones	USA	31.5.1913	Cambridge, Mass.
4:12.6	Norman Taber	USA	16.7.1915	Cambridge, Mass.
4:10.4	Paavo Nurmi	FIN	23.8.1923	Stockholm
4:09.2	Jules Ladoumegue	FRA	4.10.1931	Paris
4:07.6	Jack Lovelock	NZL	15.7.1933	Princeton, N.J.
4:0.6.8	Glen Cunningham	USA	16.6.1934	Princeton, N.J.
4:0.6.4	Sydney Wooderson	GBR	28.8.1937	Motspur Park
4:06.2	Gunder Hagg	SWE	1.7.1942	Gothenburg
4:06.2	Arne Andersson	SWE	10.7.1942	Stockholm
4:04.6	Gunder Hagg	SWE	4.9.1942	Stockholm
4:02.6	Arne Andersson	SWE	1.7.1943	Gothenburg
4:01.6	Arne Andersson	SWE	18.7.1944	Malmo
4:01.4	Gunder Hagg	SWE	17.7.1945	Malmo
3:59.4	Roger Bannister	GBR	6.5.1954	Oxford
3.58.0	John Landy	AUS	21.6.1954	Turku, Finland
3:57.2	Derek Ibbotson	GBR	19.7.1957	London
3:54.5	Herb Elliott	AUS	6.8.1958	Dublin
3:54.4	Peter Snell	NZL	27.1.1962	Wanganui
3:54.1	Peter Snell	NZL	17.11.1964	Auckland
3:53.6	Michel Jazy	FRA	9.6.1965	Rennes
3:51.3	Jim Ryun	USA	17.7.1966	Berkeley, Calif.
3:51.1	Jim Ryun	USA	23.6.1967	Bakersfield, Calif.
3:51.0	Filbert Bayi	TAN	17.5.1975	Kingston, Jamaica
3:49.4	John Walker	NZL	12.8.1975	Gothenburg
3:49.0	Seb Coe	GBR	17.7.1979	Oslo
3:46.31	Steve Cram	GBR	27.7.1985	Oslo
3:44.39	Noureddine Morceli	ALG	5.9.1993	Rieti

Table 1.2 The world record for the mile

2.5: Which information is provided by the above table? Is it possible to construct a linear model of the relationship between date and the world record? If it is possible, then construct such a model.

2.6: Give a well-founded estimate on the world record time in 2010, 2020 and 2030.

3 Marta and Marius

Martha and Marius go to a Sixth Form college. Martha loves to read. Next to her desk she has piled 5 books from the library. Each week she turns in 4 of these books back to the library and at the same time she loans 4 new books.

3.1: Draw a graph which shows the development of the number of books in Martha's pile over a 10 week period, and read off that graph the number of books in Martha's pile after 8 weeks.

Martha and Marius would like to watch the European championship in handball. They discuss whether to buy or rent a television. In order to illuminate the problem they decide to construct a mathematical model.

3.2: Which elements do you think must enter into such a model? Try to construct a model which serves to help Martha and Marius to decide whether to buy or rent.

While Martha works hard and meticulously on her homework, Marius is somewhat lazy. On average, Martha and Marius receive 2 new homework assignments each week. And, on average, Marius submits 2 assignments every other week. The school year spans over 40 weeks and hereafter every assignment must be submitted.

3.3: Construct a model which describes Marius' submission of assignments. How many assignments is Marius short after 10 weeks? 20 weeks? 40 weeks?

In their winter break, Martha and Marius work at a winter sport hotel in Sweden. The hotel manager Mats Matte has become aware that Martha and Marius have knowledge of mathematical modeling. He asks them to construct a model which clearly can illustrate and calculate how many guests take lodgings at the hotel at a given time in a busy period of 14 days. Mats informs that, at the beginning of the period, 220 the hotel has 220 guests. The 4 first days 40 guests check out each day and 10 new guests arrive. On the 5th day and forward 20 guests check out each day. On the 5th and 6th day 30 new guests arrive, and on the 7th day and forward 50 new guests arrive each day.

3.4: Give Martha and Marius an example of how they could construct a mathematical model of how many guests take lodgings at the hotel in the 14 day period.

When situations like the above are to be analyzed by means of mathematical models, there are several possibilities for constructing a model. A model can be constructed as a

- Verbal description
- An equation
- A graph
- A table

3.5: For each of the above mentioned forms of representation, discus the benefits and drawbacks and provide examples of application of the different forms of representation.

4 Population growth

In order to allocate funds to the military, transport, schools, health sectors etc., it is important for officials and political decision makers to have knowledge of a population size in a country.

The interest in the development of the population size began during the 18th century. At that time focus was directed at the ratio between a population size and the consumption of natural resources. In 1798, the British economist Thomas R. Malthus (1766-1834), published the book *Essay on the Principle on Population*, in which he presented a mathematical model for the growth of a population. Malthus' model can be given a symbolic description in terms of this formula:

$$P_t = P_o \cdot (1 + r)^t$$

Here P_t is the population size at time t, P_o is the initial population size, r is the annual growth rate (e.g. if the annual population growth is 1.5% the growth rate would be r=0.015), and t is the number of years after the start time.

4.1: The book Mathematical Modeling in the Secondary School Curriculum is an American study book in mathematical modeling and it contains an assignment in which it is informed that the US population in 1950 was 150.697.000 thereafter the US population in 1980 and 2000 must be calculated by means of Malthus' model assuming that the annual population growth is 2%. Solve this assignment.

4.2: According to the US Census Bureau, Population Division the US population in the beginning of 2000 was 274.338.367. Compare this to the results from 4.1 above. What could be the reason for the discrepancy, and is there any way to improve how to determine the population size in 2000?

5 Population growth in England and Wales

When population growth is to be analyzed by means of mathematical models there will usually be three possible mathematical methods by means of which we can approach the subject at hand: *numerical, graphical* and *symbolic*.

The numeric method is typically applied when we find data in a table like the one shown below. This table shows the population size in England and Wales between 1801 and 1911:

Year	1801	1811	1821	1831	1841	1851	1861	1871	1881	1891	1901	1911
Mio.	8.89	10.16	12.00	13.9	15.91	17.93	20.07	22.71	25.97	29.00	32.53	36.07

A very simple method to approach such a table is to determine the difference or quotient between two, on each other following, columns.

If the *difference* between two, on each other following, columns is constant for each column, then the population growth is *linear*.

If the *quotient* between two, on each other following, columns is constant for each column, then the population growth is *exponential*.

5.1: Determine whether the population growth in England and Wales between 1801 and 1911 is linear or exponential.

Graphical methods may aid our understanding of the relationship between two quantities and it allows us to clearly communicate that relationship. A graph provides an overview of how the population size increases and it is possible to read off the size in a given year.

5.2: Draw a graph of the population size in England and Wales in the period 1801-1911.

Finally we have the symbolic method. Here we arrange mathematical expressions, formulas, in which the symbols represent the quantities that figure in the model. An example of the usage of a symbolic method is Malthus' formula for the population size. The mathematical expression can be used to calculate the population size at a given time, or to calculate how much the population size changes over a given period.

5.3: Construct a mathematical expression which describes the relationship between the population size in England and Wales and the time over the period from 1801 to 1911.

Since each method provides a unique perspective on a situation, mathematical models normally involve all three methods. The three methods should also be supplemented with explanations and reasons should be given for how the model is constructed and applied.
Take a look at the formula which Malthus presented in 1798 (in exercise 4 above). You can see that he thought that the population grows exponentially. In his book he wrote that the population grows exponentially whereas the supply of natural resources grows linearly. According to Malthus this poses a serious problem: he thought that the world population grows so fast that after one century a shortage of natural resources would result in famine.

On the basis of his model, Malthus recommended that the birthrate should be reduced by delaying matrimony. Malthus feared, however, that such an initiative would result in the deterioration of morality because it would lead to more premarital sex.

As we now know, Malthus was not exactly right. But even though his theory was incorrect, he did direct our attention on the fact that population growth and the growth of natural resources can be described by means of mathematics.

5.4: Consider why Malthus thought that the population grows exponentially whereas the supply of natural resources grows linearly. Describe the factors which affect population growth, and describe the factors which affect the growth of the supply of natural resources. Use these considerations to suggest which kind growth we are dealing with when we talk about the growth of the supply of natural resources.

6 Alcohol degradation

It can be a great advantage to know the amount of blood alcohol content at a given time after the consummation of alcohol. To this end, Swedish chemist Widmark developed a mathematical model for the blood alcohol content (measured in alcohol pro mille blood). Since alcohol is water soluble, it can only be distributed in the body's water. Therefore, if one wants to calculate the blood alcohol content, one must first ascertain how much of a person's body mass is comprised by water. Widmark developed a so-called reduction factor r, with which one can calculate how much water one's body contains. This factor is gender specific:

$$r_{Male} = 0{,}3161 - 0{,}0048 \cdot v + 0{,}0046 \cdot h$$
$$r_{Female} = 0{,}3122 - 0{,}0064 \cdot v + 0{,}0045 \cdot h$$

Here v denotes the person's weight in kg, and h denotes the person's height in cm.

6.1: Using the above formula, calculate your reduction factor.

Widmark thought that if a person's reduction factor is know it would be possible to calculate that person's blood alcohol content from this formula

$$C_t = \frac{n \cdot D}{r \cdot w} - \beta \cdot t$$

Here
Ct: the blood alcohol content (measured in grams of alcohol per liter blood) at time t.
n : the amount of standard units of alcohol drunk by the person.
D : the amount of alcohol in a standard unit of alcohol in grams (a standard unit contains 12 grams of alcohol).
r : the person's reduction factor.
w : the person's body weight in kilos.
β : metabolic rate in grams per liter per hour (for males: 0.18; for females: 0.15).
t : the time in hours.

6.2: Draw a graph which describes the development of your blood alcohol content if you consume 3, 5, and 8 units of alcohol. What is your blood alcohol content in each of the three cases after 4, 6 and 8 hours?

(Continue on the next page)

6.2 (...continued)

6.3: Malthe weighs 80 kilos and he is 178 cm high. Today he was pulled over by the police and asked to breathe into an alcoholmeter. His blood alcohol content was measured to be 0.93. Malthe explained that he had his last drink 3 hours ago. How many units of alcohol did Malthe drink 3 hours ago if he is telling the truth?

7 Medicine in the blood

When we ingest medicine it gradually breaks down in the body. In order to describe this process, we use a model in which it is assumed that over a given period a given percentage of the medicine is broken down. For instance, it is reckoned that when a person ingests aspirin, about half of the medicine will be broken down after 30 minutes.

7.1: Assume that you ingest 750 mg aspirin.

(a) How much aspirin is left in your blood after 4 hours?

(b) Construct a symbolic model which can be used to calculate the amount of aspirin in the blood at a given time.

7.2: Draw a graph which describes the relationship between the amount of aspirin in the blood and the time.

A lot of people regularly ingest medicine. Assume that you ingest 200 mg aspirin every 4 hours.

7.3: What is the long term effect (e.g. over a period of 4 days) of this regular ingestion? E.g. construct a table and draw a graph which describes the relationship between time and the amount of aspirin in the blood.

7.4: Look at the graph you drew in the last exercise. Which type of development do you think we are dealing with here? Compare this with what you know about the degradation of alcohol in the body. Does alcohol and medicine break down differently?

8 Yeast cells

In order to make wine, a natural fermentation process is exploited. When the grapes are crushed, yeast cells on the surface of the grapes are mixed with the fruit juice within the grapes. In the fermentation process the yeast cells convert sugars to alcohol (ethanol) and carbon dioxide.
In fermentation yeast cells run through different phases. In each phase the development of the amount of yeast cells is different.

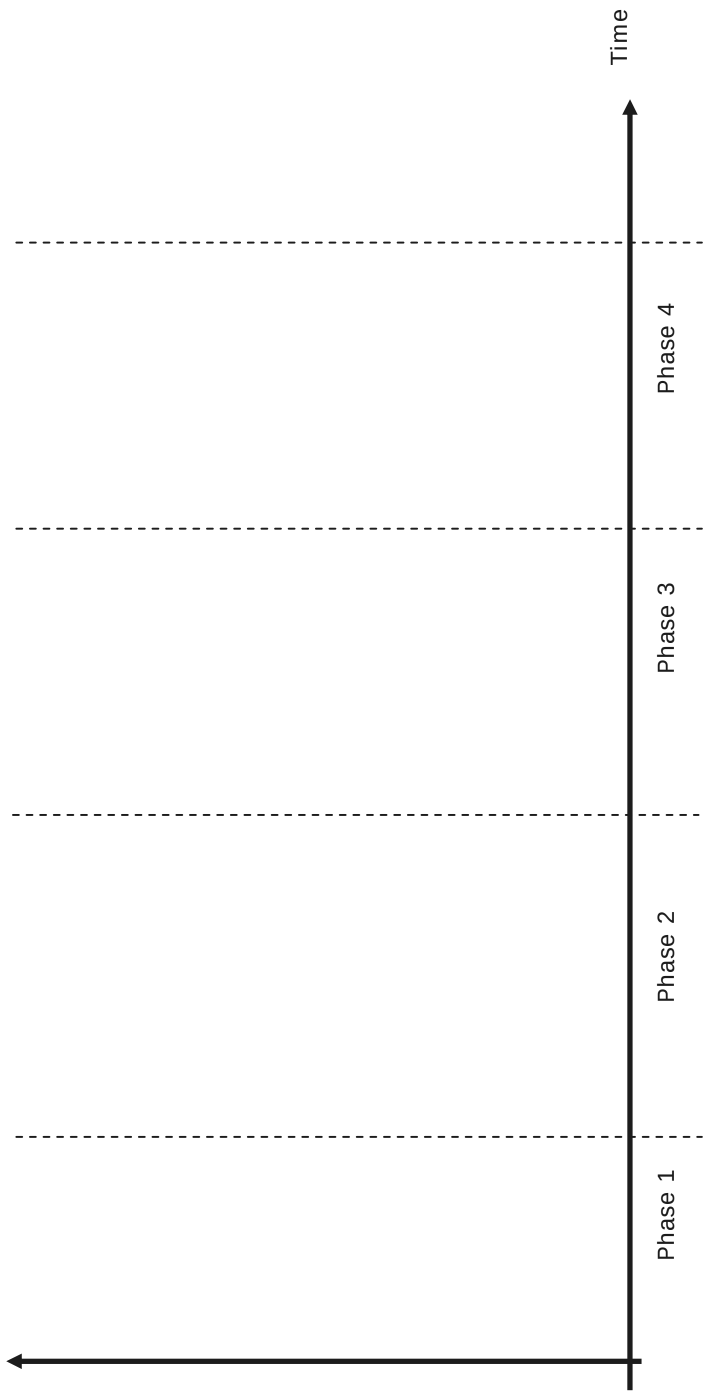
The *initial phase*: In the initial phase the yeast cells have to adapt to their new environment. In this phase almost no cell divisions take place. When the yeast cells have adapted to their environment they go into the second phase.

The *exponential phase*: In the second phase the yeast cells divide at a constant rate. At some time the cells have divided themselves so many times that the sugar concentration cannot support that many yeast cells. At this time the yeast cells pass into the third phase.

The *stationary phase*: In the third phase the number of dying yeast cells is nearly equal to the number of cell division. In all these phases the living yeast cells have convert sugars to alcohol (ethanol) and carbon dioxide, but alcohol is poisonous for the yeast cells and at some time so much alcohol has been produced that the cells can no longer survive.

The *death phase*: In the final phase the yeast cells gradually begin to die. Here far more cells die than there are cell divisions.

8.1: Draw a graph which describes the growth of yeast cells in the different phases.



8.2: The concentration of alcohol depends on the amount of yeast cells. The more yeast cells, the more alcohol is converted. Each yeast cell converts a fixed amount of sugar to alcohol and carbon dioxide. In the previous exercise you drew a graph which describes the growth of yeast cells in the different phases. Use this description to consider how the alcohol concentration develops over time. Draw a graph which describes how the alcohol concentration growth in a fermentation process.

In the table below you can see the results from an experiment in which the alcohol concentration in a fermentation process has been measured frequently.

Hours	5	10	15	20	25	30	35	40	60	80	100	120	140	200	250
Alcohol %	0,3	0,4	0,5	0,7	0,8	1,0	1,3	1,6	4,0	6,5	9,3	11,1	12	12,5	12,5

Source: http://w2.ef.dk/netbog/Htxopg/kap32.htm

8.3: **(a) Compare these results with the graph which you drew in the previous exercise. Does your graph match these numerical data?**

(b) Look at the results from the first 40 hours of the experiment. Examine whether the growth of alcohol percentage between 5 hours and 40 hours is exponential or linear.

The growth of alcohol percentage belongs to a type of growth called logistic growth.

8.4: Use what you know about the growth of alcohol percentage to describe what characterizes logistic growth.

8.5: Numerous things grow logistically. Find some examples of things which grow logistically. Describe which factors could be responsible for these things growing logistically.

9 Tsetse flies

The African Hilus tribe earns a living from cattle breeding. The tribe’s income depends on how much cattle can be sold each year. The greater the tribe’s cattle herd, the greater the tribe’s income. Since it seldom rains where the cattle grazes, the tribe has constructed a well and an irrigation system. Satisfied they observe that the pastures turn more and more fertile as it is irrigated. Further, they observe that the herd size grows as the pastures become more fertile. The tribe has now learned that there is a relationship between the fertility of the pastures and the herd size: The less grass growing of the pasture, the smaller the herd.

Consequently, the tribe chooses to irrigate the pasture even more frequently. But the repetitive irrigation has an unpleasant side-effect: the amount of the feared tsetse flies begin to grow very quickly. The more moist the pasture, the more do the tsetse flies breed. And since tsetse flies can infect the cattle with the often fatal sleeping sickness, the tribe is concerned that the cattle size is diminished.

9.1: Attempt to sketch the mentioned relationships in a way which at one glance provides an overview of the most important factors.

10 Conclusion of the theme 'growth'.

Write a short summary of your activities in connection to this theme. The summary must end with a conclusion on what you take to be the most important results of your work on this theme – i.e. which mathematical properties of different types of growth functions have you found. You can further put forward criticism as well as questions regarding the theme and the course. You can finally suggest changes etc.