Education and Culture
Lifelong learning programme
COMENIUS

Teaching Material

Suggestion

Introduction of Fermat's Principle

The life guard problem describes a lifeguard's difficulty to find the quickest way, to get from his place at the beach to the person in water who is in need –assuming the lifeguard is looking at the "situation" from the side. As running along the beach is faster than swimming, the quickest way is not to go straight into the water. The quickest way does not lead in a straight way to the drowning person therefore you are going to walk on the beach for a longer time before you start swimming. Fermat's principle shows that light behaves like the perfect lifeguard. On account of its vividness this example is very well suited for introducing the topic. Main focus lies first of all on the mathematical minimising problem.
You may assume that because of the unknown analysis very few to no extreme value problems have been dealt with yet. Therefore the introduction should be lead very tightly. This can be done by a step by step calculation of the time the lifeguard needs for a certain way, chosen by the pupil (see worksheet 1). By doing that, the probably new contents of Pythagoras' Theorem will be practised again. Then attention is drawn to the flexible position where the lifeguard enters the water and the target function describing the time the lifeguard needs will be posted. While looking at the graph of the target function, its statement will be considered and therefore a need for the calculation of the low point will be created.

Checking the results of the first work plan can be carried out within the whole group. It can also be carried out by presenting the solutions e.g. on a slide or piece of paper so that pupils can check individually if the class is familiar with this procedure (see solutions for worksheet 1). In this case the pupils may compare results of the subtasks or if they have problems they may pick up a part of the solution and go on with it. Particular solutions could be put up in various places in the classroom so that a complete examination will be made difficult. This method could be differentiated by giving those pupils, who have difficulties, written advice on how to get to the solution before they may look at the actual solution. These solutions might be put up as well.
In a second part the pupils will have another opportunity to "walk the path" to get to the solution. Especially those pupils who have not been able to solve the first part on their own now have the opportunity to check directly whether they have understood the solution step by step as they are now in a situation to solve a similar exercise on their own. Fermat's Principle is presented to the pupils in the form of "Light acts like a perfect Lifeguard". In addition to the speed of light in air and in water they will receive a description of an experiment with a source of light (laser) in the air and an object under water. The exercise is to calculate the point that has to be aimed at with the laser in order to hit the object (see work sheet 2). In order to verify the result a suitable experiment should not be left out (see also solutions to work sheet 2) of course.
In addition to both work sheets mathematical and physical discussions within the class plenum should follow: Regarding Mathematics the procedure may be discussed again in a structured way in order to develop a kind of "recipe" for extreme-value-problems, which will occur later on. Regarding Physics this very calculation should be generalized. Hence refraction can be seen as a result of a beam of light meeting with objects that have different

densities. It can be used to explain phenomena occurring in daily life when looking into water.
However, it is also important to debate the conclusion of the reflection attribute “angle of incidence equals angle of reflection”.

Material needed

The experiment is needed only once within this unit since it is needed for controlling the calculations only. Necessary material for the experiment setup (see pictures)

- 1 glass envelope (at least 40 cm long and 30 cm high),
- 1 laser pointer,
- 1 “worm” or comparable equipment,
- tripod,
- measuring tapes

Work Sheets (to copy – see next pages)

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The **ScienceMath**-project: **Fermat meets Pythagoras**
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WS 1: The Lifeguard-Problem

Lifeguard Mitch is standing in front of his tower when he views a person in water who is in distress. The direct way to the water is 50 m. From there it is another 50 m straight on and then 50 m south to get to the person in need (figure 1). Mitch knows that he needs 7m/s ashore and only 2m/s in the water. In order to get to the person as quick as possible he starts off by running straight along the shore to a point Q from where he swims directly to the person. On the shore he covers a distance of u meters and in w meters in water.

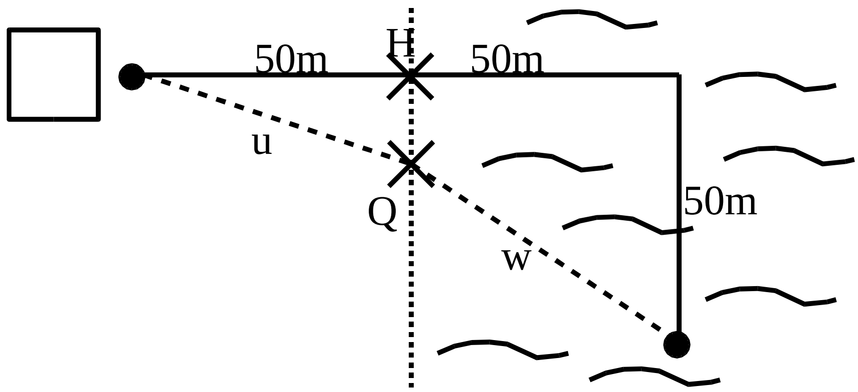


Figure 1

a.) How long would it take if Mitch runs 50 m directly to the water first (point H) and then swims a distance of 71 m to the person?

b.) He could also run to the point from which he could swim in a straight line to the person. This is equivalent to 71 m ashore and exactly 50 m in the water. How long will it take him?

c.) Imagine you are the lifeguard. Pick any point Q (pick really quickly) between the both extremes from a.) and b.), choose this point as you think will be fastest to the person.
Calculate the lengths of the distances u and w (see picture 1) as well as the necessary times t_u and t_w. Then figure out the overall time t_{entire} you need to reach the person. Compare with your neighbours. Who of you is the quickest?

d.) A lifeguard has chosen point Q in such a way that it is situated in a distance x to point H. Calculate t_{entire} in accordance to x.
Hint: Calculate step by step the same numbers as in c.) however it should be in accordance to the variable x.

e.) The result from d.) is a function term t(x). Consider the chart of the function on the computer accordingly. Chose an appropriate coordinate system. Write down what you can see from this chart in general and give some examples.

f.) Determine the distance x in such a way that the lifeguard is quickest.

Solutions

a.) To get to the shore he needs (50m) : (7m/s) ≈ 7,1s. To get to the person, he needs (71m) : (2m/s) ≈ 35,5s. Altogether he needs 42,6s.

b.) This way he would need (71m) : (7m/s) ≈ 10,1s on dry land and (50m) : (2m/s) = 25s in the water. Altogether 35,1 s.

c.) <u>Example: point Q is 10m away from H</u>

According to Pythagoras' theorem $u = \sqrt{(50^2 + 10^2)} \approx 51$, therefore u is approximately 51m long. The length w can also be calculated by Pythagoras' theorem. As one knows the downward length is 50m, point Q is 10m away from H it has a 40m distance from that point on the shore which with Q and the person completes a right-angled triangle. Valid therefore is $w = \sqrt{(50^2 + 40^2)} \approx 64$; the length w is approximately 64m.

Hence it follows that: t_u=(51m) : (7m/s) ≈ 7,3s, t_w=(64m) : (2m/s) = 32s, the entire time needed is consequently 39,3s.

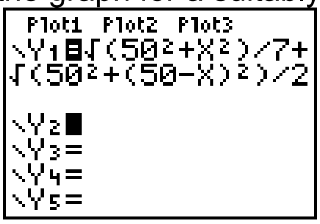
d.) Again the lengths u and w are concluded from Pythagoras' theorem:

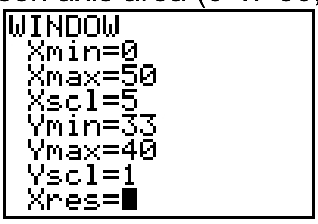
$u = \sqrt{(50^2 + x^2)}$ and $w = \sqrt{(50^2 + (50 - x)^2)}$.

Hence it follows that: t_u= $(\sqrt{(50^2 + x^2)}m) : (7m/s)$, t_w= $(\sqrt{(50^2 + (50 - x)^2)}m) : (2m/s)$, the entire time needed is therefore

t_{entire}=[$(\sqrt{(50^2 + x^2)}m) : (7m/s)$]+[$(\sqrt{(50^2 + (50 - x)^2)}m) : (2m/s)$].

e.) Input of the function on the graphic calculator is done as shown on fig.1. Fig.2 shows the graph for a suitably chosen axis area (0<x<50, scale unit 5, 33<y<40, scale unit 1)





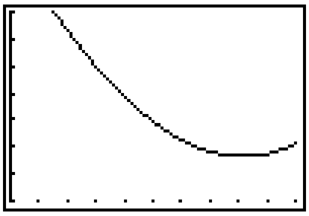
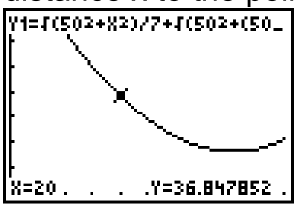


figure 1: function input, *figure.2a: chosen axis area,* *figure 2b: accompanying graph*

This graph shows the time t (y-point) the lifeguard needs if he chooses point Q with the distance x to the point H. E.g.:



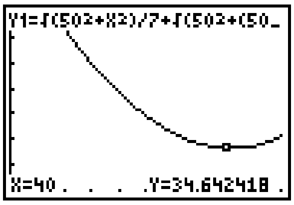


figure .3a *figure .3b*

figure 3a: If Q is 20m away from H the lifeguard needs approximately 36,8s.
figure 3b: If Q is 40m away from H the lifeguard needs approximately 34,6s.

f.) The minimum on the image can be calculated by using a computer. As seen in fig. 4, the lifeguard is fastest if he chooses point Q along the shore approximately 40,815m away from point H. He then needs approximately 34,6s.

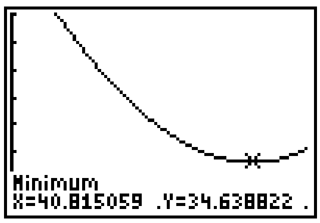


figure 4: The lowest point on the curve was determined by computer.

WS 2: Fermat's Principle

Light does not always travel at the same speed. The speed of light in air is 300.000 km/s whereas the speed of light in glass and water is only about 200.000 km/s and 225.000 km/s, respectively. As well as that, light always acts as a perfect lifeguard. This means, a beam of light always chooses the shortest way possible to get from a point A to a point B. This behavior of light is called Fermat's Principle - named after Pierre de Fermat (1608-1665) who was the first to discover this.

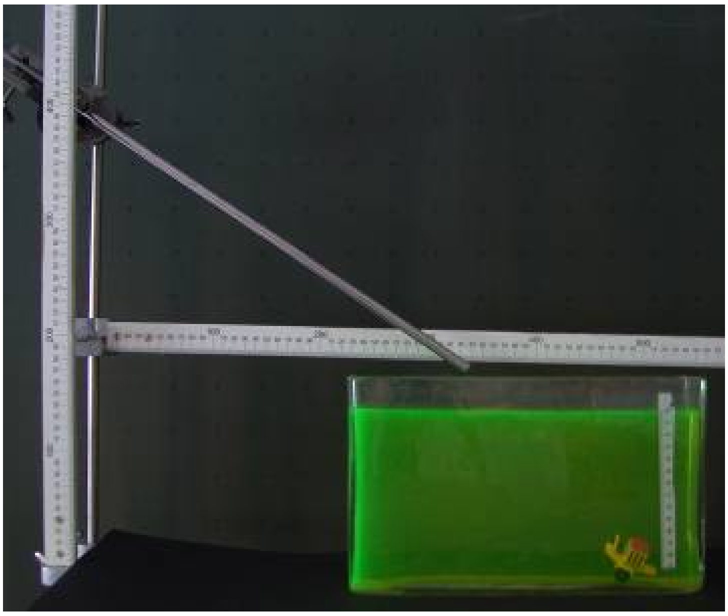


Figure 1: The snail in the water.

Task:
A small water snail wants to have some light in its house. In order to achieve this, the light of a laser placed outside of the water needs to go straight through the "roof" of the snail's house (see figure1).

a.) If we go beyond the arrangement in figure 1 and transfer it into a Cartesian coordinate system we get the laser in point L(0/40), the water's surface along the straight line g with the equation y=14 and the roof of the snail's house in point S(50/3) (all measurements in centimeter!). Draw a coordinate system and enter L, g and S (choose a suitable scale).

b.) The laser can only be placed in such a way that its light meet the water in point Q(x/14). Place a random point Q in your coordinate system designed in a.).

c.) Where does point Q have to be in order for the light to travel the fastest way from L to Q and from Q to S?

d.) With the help of Fermat's principle, consider what will happen if the laser is pointed to the point you calculated in c.). Also consider what will happen if a different point as calculated in c.) is pointed at.

Solution for worksheet 2
a.) and b.)

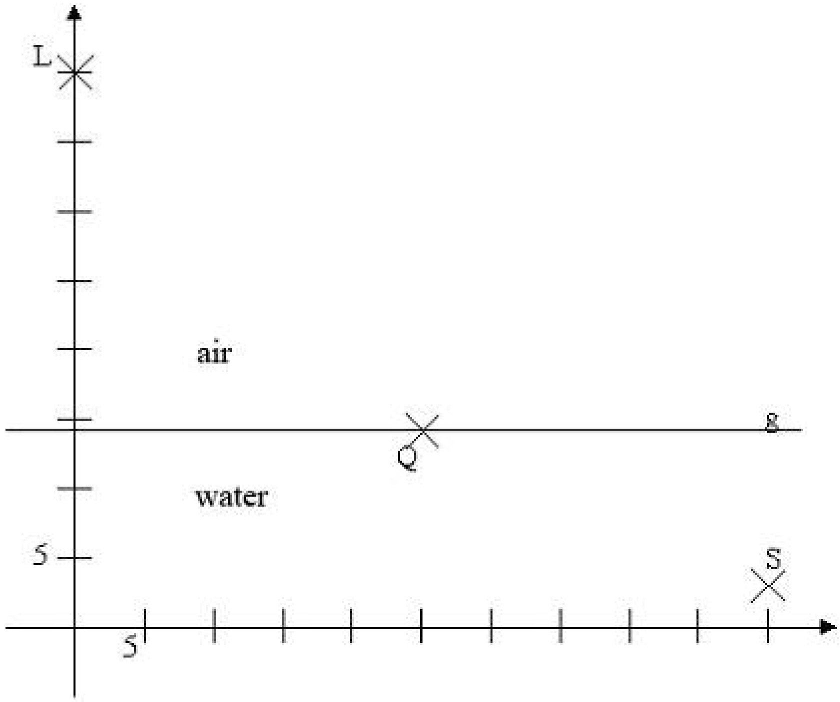


figure.2: Solutions for a.) and b.)

c.) With the help of the guide lines and quantities shown in figure. 3 the solution becomes more comprehensible:

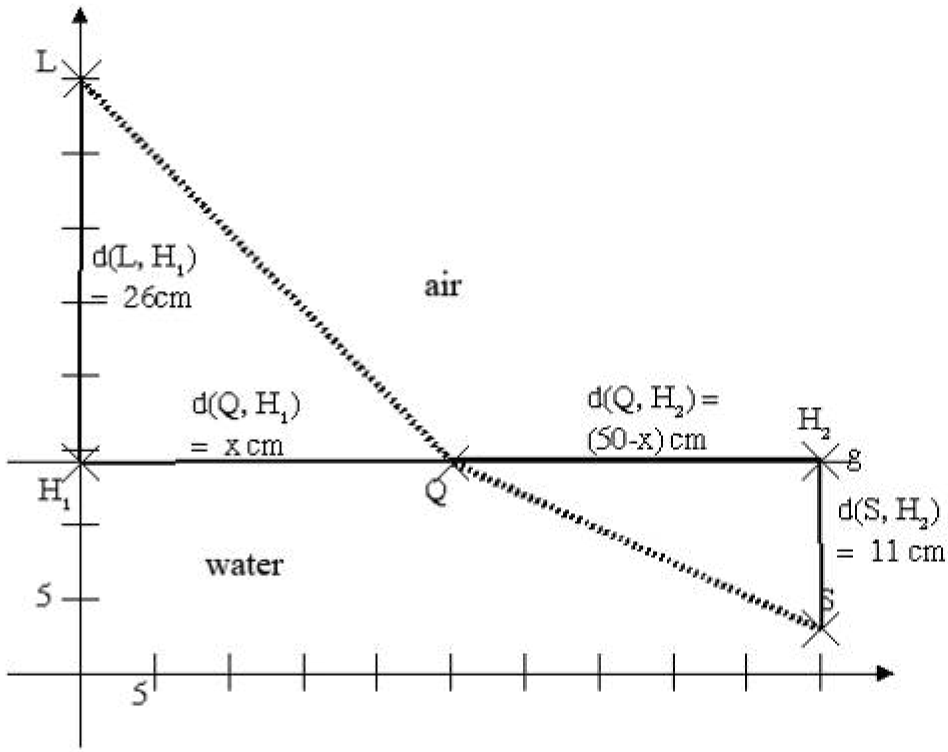


figure. 3 suitable guide lines and quantities

The way from L to Q is $\sqrt{26^2+x^2}$ long. In air the light travels at the speed of 300.000.000 m/s = 30.000.000.000 cm/s. Hence the light needs the time t_1 = $\sqrt{26^2+x^2}/30.000.000.000$ seconds on its way from L to Q.
The way from Q to S is $\sqrt{11^2+(50-x)^2}$ long. In water the light travels on a account of its speed in water (225.000.000 m/s = 22.500.000.000 cm/s) t_2= $\sqrt{11^2+(50-x)^2}/22.500.000.000$ seconds.
As a result from the required time t_{requ} the light needs to get from L to S via Q the dependency t_{total} (x)= $\sqrt{26^2+x^2}/30.000.000.000 + \sqrt{11^2+(50-x)^2}/22.500.000.000$ (in seconds) emerges.

On the computer the picture of this function t_{requ} can be seen only in a suitably chosen area. In figure 4 0 < x < 50 as well as 0,000000001 < y < 0,000000003 were chosen.

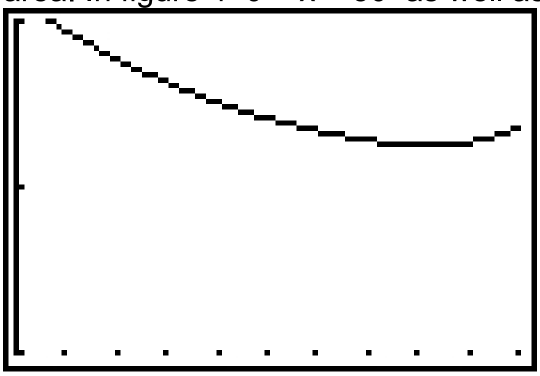


figure.4: picture of the function $t_{requ.}$

<u>The calculation of the low point - with the help of the computer - is approximately **Q(41/14)** for the quickest way.</u>

d.) If the laser is pointed roughly on the point Q(41/41) the „roof“ of the snail’s house will be hit at point S (see figure 6a). At every other point Q the beam of light “loses it was” in such a way that point S will not be reached and the snail’s house remains in the dark (see figure 6b).

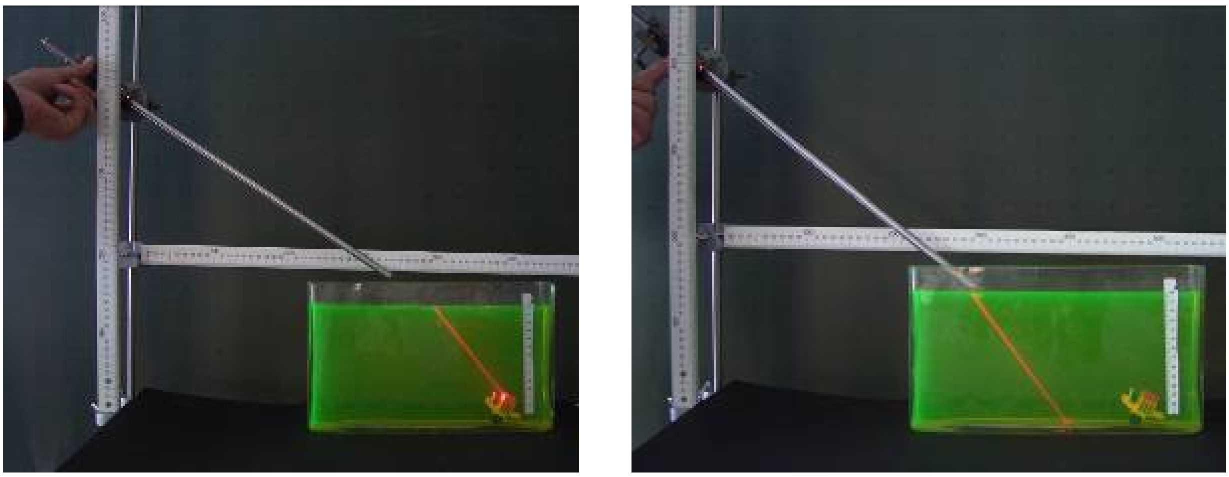


figure.6 The snail’s house is either illuminated or stays dark.