The **ScienceMath**-project: **Centre of Gravity concept**
Idea: Astrid Beckmann,
University of Education Schwaebisch Gmuend, Germany

Teaching Material

Suggested Lesson Plan

For implementation in class it is suggested to do work along different stations which cover the topic of centre of gravity/point of intersection of the medians in a triangle. The stations have to be designed in such a way to make the students grasp the concept of the centre of gravity in technical-physical terms, but at the same time, contain a mathematical model. The suggestions made here take this aspect in consideration.
In order to link the various aspects of the centre of gravity concept, which the students will experience at the different stations, each group ought to produce a poster for a final joint presentation in class. It is also conceivable to extend the work through a web quest on the centre of gravity topic, in which further information could be added to the final presentation.

At higher levels, the centre of gravity in planes and bodies could additionally be worked out mathematically, e.g. with the help of integral maths, if the limiting function f (x) is known.

In addition to this, the topic can be extended through experiments of physical phenomena, showing connections between stability, position of the centre of gravity and supporting constructions.

Stations (cf. following pages):

- **station: suspension methods**
- **station: weighing methods**
- **station: symmetry axes**
- **station: planes**
- **station: mathematical definition of points of gravity using a computer**
- **station: carton planes**

Potential extension, respectively, alternatives for upper secondary level:

- **Worksheet: centres of gravity in planes and bodies**

Potential extension:
Connection between stability and position of centre of gravity

- **Stations: Phenomena**

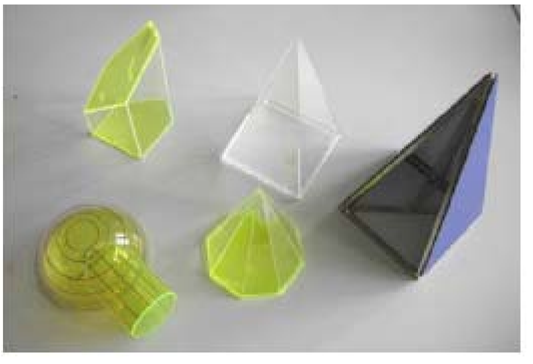
Station: Suspension Method

Content and objective:
Identifying the centre of gravity in a variety of bodies by the suspension method

Material:
A variety of bodies, if possible with provisions for hanging/threads,

Duplo blocs are well suited for this, as one can compose a number of different bodies easily, and the thread for hanging can be clamped between two blocs. Hollow bodies or those made of clear plastic are a great advantage, since the centre of gravity can also be established inside the bodies using straws, that can be freely attached to the mounting bracket. On the other hand, measuring students' private objects can be very motivating.

Thread, ruler, adhesive tape, pencil or felt pen, straws for marking the lines of gravity.

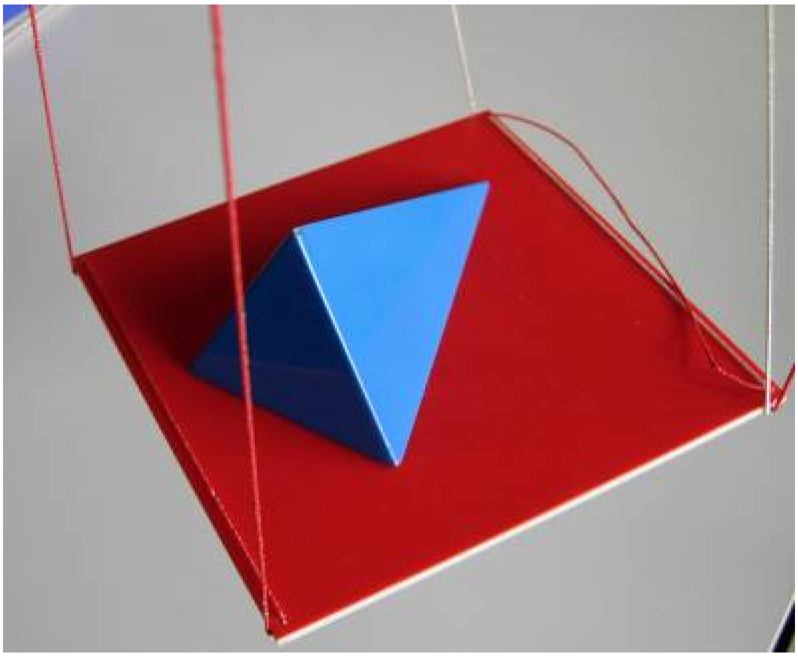


Realisation:
The bodies are hung up in various (at least two) positions; the lines of gravity are drawn on each of them and marked with adhesive tape; the centre of gravity is determined as the point of intersection of the lines of gravity.

Station: Weighing Method

Content and objective:
Establishing the centre of gravity of various bodies by the weighing method

Material:
Various bodies, a pair of hanging scales: e.g. a board to put the bodies on, with threads attached to it for suspending, respectively, a tripod to hang it from; ruler, adhesive tape, pencil or felt tip, straws to mark the lines of gravity

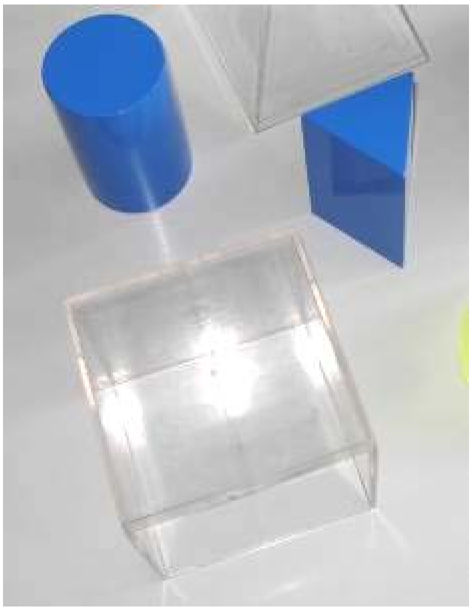


Realisation:
The bodies are put on the scales in various positions (at least two), the lines of gravity are marked on each of them, and the centre of gravity is determined by the point of intersection of the lines of gravity.

Station: Symmetry Axes

Content and Objective:
Determining the centre of gravity of various bodies by the suspension method, discovering symmetry axes as the lines of gravity

Material:
Various line symmetric bodies with more than one symmetry axis, preferably with installed hanging device (hook) at the symmetry axes; thread, ruler, adhesive tape, pencil or felt tip, respectively straws to mark the lines of gravity



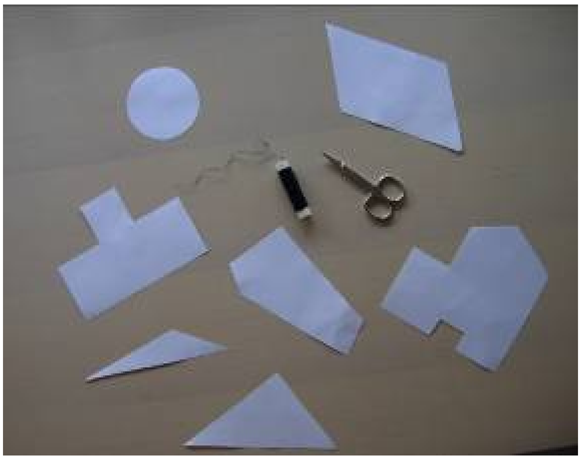
Realisation:
The bodies are hung on the hook in various positions in such a way that the symmetry axes are equal to the lines of gravity. The lines of gravity are marked in each and the centre of gravity is determined by the point of intersection of the lines of gravity.

Extension:
Various axially symmetric bodies and the hanging devices (adhesive tape, hooks etc.) are put at students' disposal. After the position of the symmetry axes is found out, the hooks are screwed in the symmetry axes. Then, the centre of gravity is established by the suspension method.

Station: Planes

Content and Objective:
Determining centre of gravity of various planes by the suspension method.
Also: Identifying symmetry axes (if given) as lines of gravity.

Material:
Various flat cardboard materials with threads for suspending them; scissors for cutting holes in the cardboard if necessary



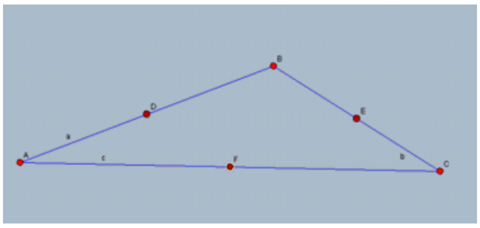
Realisation:
The cardboard planes are hung up in various positions, the lines of gravity are marked on each and the centre of gravity is determined by the point of interception of the lines of gravity.
Hint: If the planes are hung in front of a solid wall, it is easier to draw the lines of gravity.

Balancing it with the index finger confirms the position of the centre of gravity.

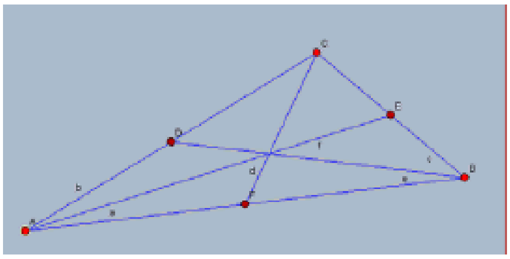
Station: Determining centre of gravity with the help of a computer (Using a dynamic geometry system)

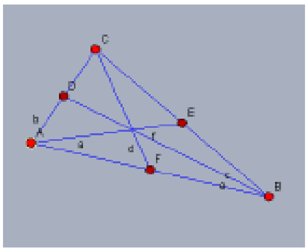
<u>Content and Objective:</u>
With the help of a dynamic geometry system, the following invariance will be discovered: In a triangle, the three medians meet in one point. This is the centre of gravity of the triangle.

<u>Material:</u>
Computer with geometry programme installed,
Task: Draw any triangle with its medians. What do you realise? (Hint: Change the triangle also by dragging one point.)
Or: prepared file with triangle and mid points of each side marked (Cf. graph, here: Cinderella).



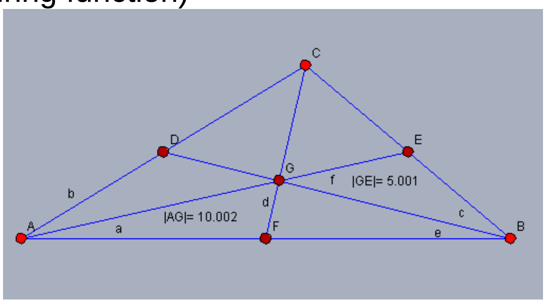
<u>Realisation:</u>
The basic construction is extended by adding the medians. The triangle is randomly modified. One can discover that the medians always intersect in one point.





<u>Extensions:</u>
Task: Print any triangle, stick it onto the cardboard and balance it on your index finger. In which point do you have to support the triangle, to keep it in balance (Centre of gravity)?

Task: Discover further relationships. In which ratio does the centre of gravity divide the medians? (Use the measuring function)

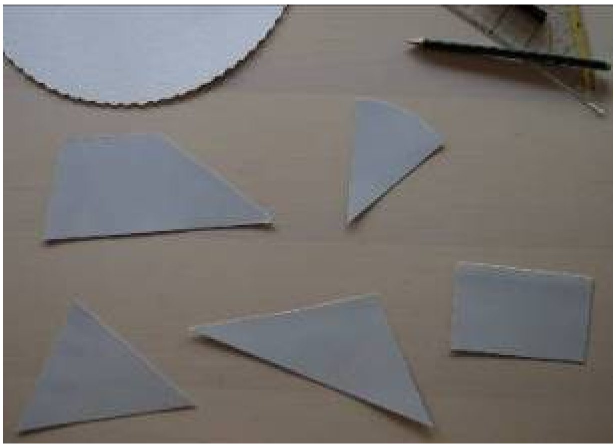


Task: Give reasons why the three medians always intersect in one point. Hint: Stretch the sides at G (centre of gravity) by factor 0.5.

Station: Carton planes

Content and Objective:
Determining the centre of gravity of various areas, describing its position and checking its correctness

Material:
Various flat pieces out of cardboard, pencil, ruler, if necessary, cardboard for cutting out further pieces



Realisation:
The centre of gravity of an area is first worked out graphically (possibly by experiment as well) on the basis of the insights gained in the previous experiments.
Its position is determined and described by the size of the areas.

For example: The centre of gravity in a triangle is the intersection of the medians. Its correctness is checked by hanging the body attached to this point or supporting it there.

For example: At its centre of gravity, a triangle can be balanced on the index finger.

Idea: Astrid Beckmann,
University of Education Schwaebisch Gmuend, Germany

Extension, respectively, alternative Suggestion for Secondary II

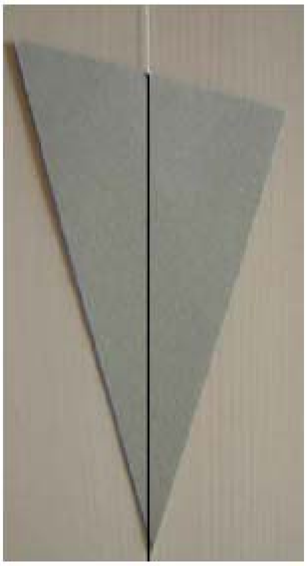
Note: You can find all tasks to this alternative suggestion on p.17 as masters that can be copied.

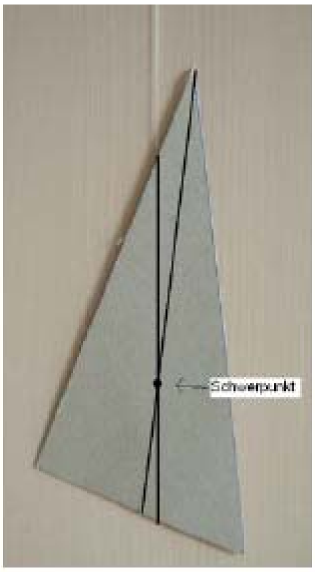
General Task:
Establish the position of the centre of gravity of a triangle, rectangle, cube, pyramid etc.

Task 1: Centre of gravity in a triangle

Material: cardboard, scissors

Task:
1. Cut a triangle out of a cardboard.
2. Enter the medians in your triangle and then balance the triangle on your index finger at the point of intersection.
   Note: What is the importance of the medians' intersection?
3. Hang the triangles successively at different points (at corners or sides) and draw the respective lines of gravity (as in the pictures below, preferably on the back where nothing is drawn). Balance the triangle on your index finger at the intersection of the lines of gravity.
   Note: What is the importance of the point of intersection of the lines of gravity?
4. Confirm the equality of both points of intersection.



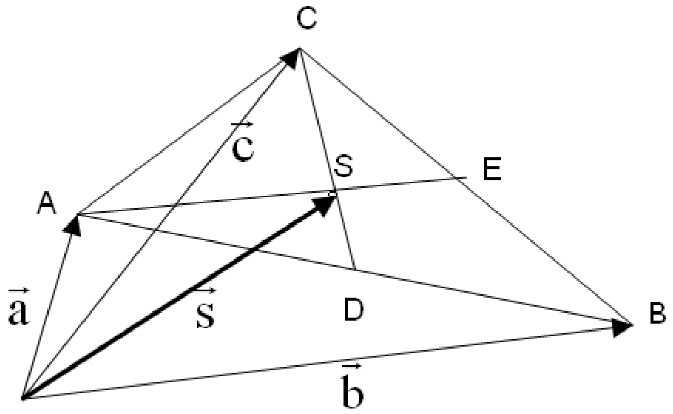


Task 2: Analytical Description of the Centre of Gravity of a Triangle

Task: Any triangle ABC can be described through the position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ of its points A, B, C.

Express the position vector $\vec{s}$ of the centre of gravity S by $\vec{a}$, $\vec{b}$ and $\vec{c}$

Solution:



The following reasoning makes use of the fact that the point of gravity S is the medians' point of intersection (here CD and AE) and that the medians are divided by S to a ratio of 1 : 2 and the sections, consequently, are 1/3 respectively 2/3 the length of the medians.

$$\begin{aligned}\vec{s} &= \vec{a} + \overrightarrow{AC} + \frac{2}{3}\overrightarrow{CD} \\ &= \vec{a} + (-\vec{a} + \vec{c}) + \frac{2}{3}(-\vec{c} + \vec{b} - \frac{1}{2}\overrightarrow{AB}) \\ &= \vec{c} + \frac{2}{3}(-\vec{c} + \vec{b} - \frac{1}{2}(-\vec{a} + \vec{b})) \\ &= \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})\end{aligned}$$

Task 3: Centre of Gravity of a Quadrangle

Material: cardboard, scissors, a prepared trapeze such like in the photo below, thread for hanging, scales (showing a minimum of 1/10 g).

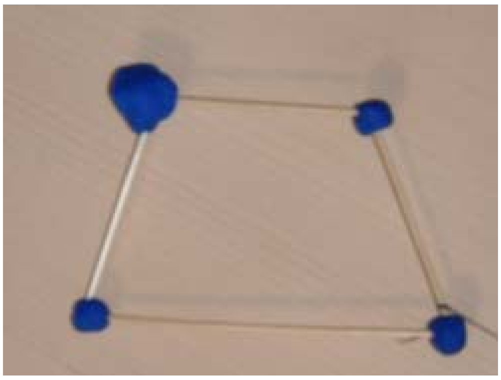


Photo: Prepared trapeze (preferably of the same measurements like in task 1), consisting of simple wooden skewers and corners made of play-dough. At least one corner should clearly differ in mass from the other corners. This can be achieved by a bigger lump of play-dough, or even better, by a small metal ball, invisibly placed inside it.

Task: Check if the following analytical description (in analogy to the triangle) of the centre of gravity is generally true in the quadrangle:

$$\vec{s} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$$

To do this, complete the following tasks:

1. Draw the following trapeze ABCD with A (0/0), B (12/0), C (8/7) and D (2/7) on the cardboard.
2. Calculate the position of the centre of gravity on the basis of the function, given above.
3. Mark the centre of gravity in the cardboard trapeze on the basis of the result.
4. Check the position of the centre of gravity in the cardboard triangle experimentally by the suspension method.
5. Check the position of the centre of gravity in the completed trapeze by the suspension method too. Estimate the approximate coordinates. What do you notice?
6. The physical centre of gravity is given through the following equation:

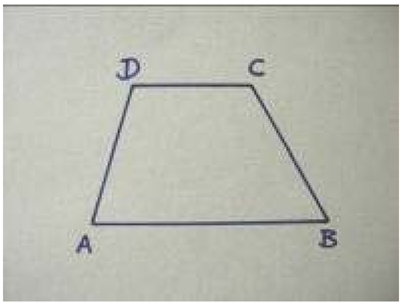
$$\vec{s} = \frac{1}{m_1 + \ldots + m_k} \sum_{i=1}^{k} m_i \overrightarrow{x_i}$$

   m_i indicates the mass of the body point i and x_i its positions.

   Explain the equation and use it to explain your observations in the experiments.
7. Answer the question above concerning the general validity of the equation for the centre of gravity of quadrangles.

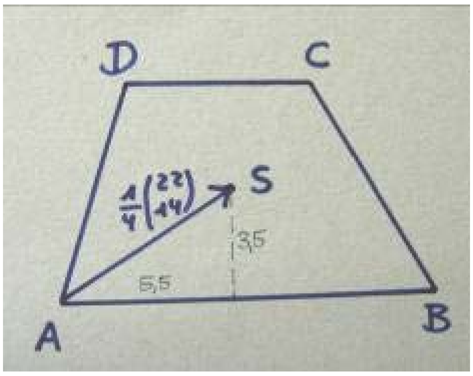
Suggested solution of task 3:

a)

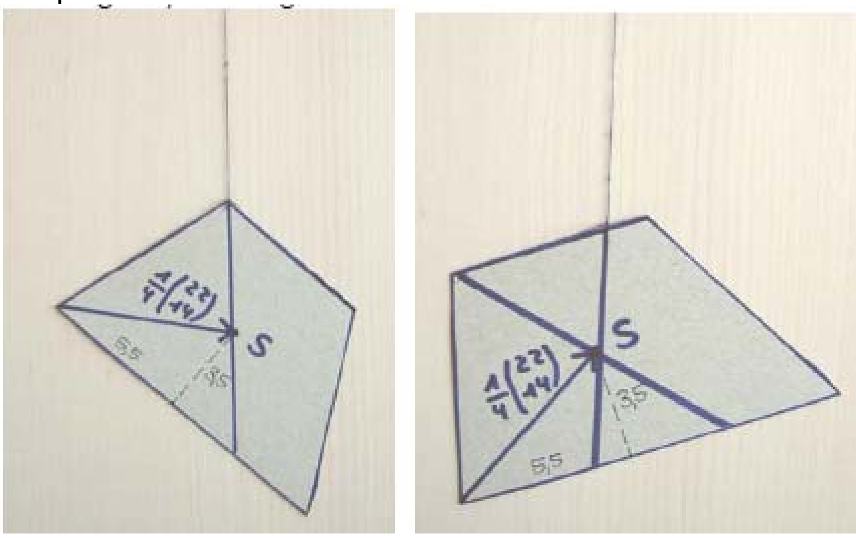


b) $\vec{s} = \frac{1}{4}(\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}12\\0\end{pmatrix} + \begin{pmatrix}8\\7\end{pmatrix} + \begin{pmatrix}2\\7\end{pmatrix}) = \frac{1}{4}\begin{pmatrix}22\\14\end{pmatrix} = \begin{pmatrix}5,5\\3,5\end{pmatrix}$, i.e. S(5,5/3,5)

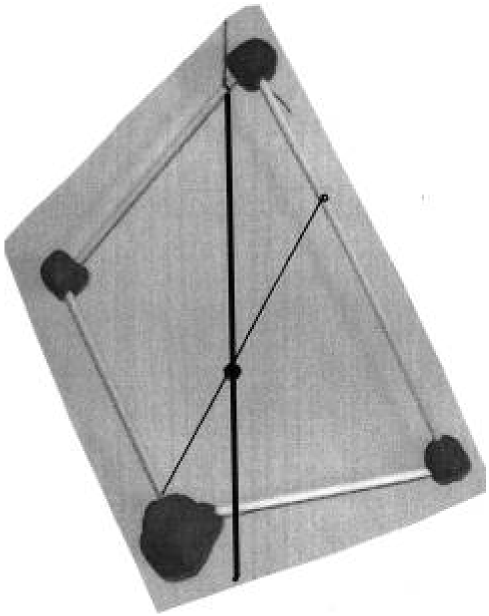
c)



d) The position of the centre of gravity is confirmed (considering experimental inaccuracies) by the suspension method.



e) The suspension method shows that the positions of the points of gravity in the cardboard trapeze and the trapeze prepared with pieces of wood and play-dough don't coincide in spite of roughly the same dimensions.
(Clue: In the different hanging positions, the lines of gravity can be drawn on a piece of paper held behind the trapeze in each case.)



Here, the position of the centre of gravity is roughly (4.7/4.0).

f) Comparing given physical formula and experiments done, an illustration of the formula can be shown best by the mass distribution of the trapeze. This can approximately be done by weighing the corners, neglecting the weight of the sides (wooden connections).
Example: The play-dough balls at corners A, B and C have masses $m_1 = m_2 = m_3 = 0.1$ g, the ball at corner D has mass m_4= 0.2 g.
Therefore:

$$\vec{s} = \frac{1}{0,1+0,1+0,1+0,2}(0,1\begin{pmatrix}0\\0\end{pmatrix}+0,1\begin{pmatrix}12\\0\end{pmatrix}+0,1\begin{pmatrix}8\\7\end{pmatrix}+0,2\begin{pmatrix}2\\7\end{pmatrix})$$

$$= \frac{1}{0,5}(\begin{pmatrix}1,2\\0\end{pmatrix}+\begin{pmatrix}0,8\\0,7\end{pmatrix}+\begin{pmatrix}0,4\\1,4\end{pmatrix}) = 2\cdot\begin{pmatrix}2,4\\2,1\end{pmatrix}=\begin{pmatrix}4,8\\4,2\end{pmatrix}$$

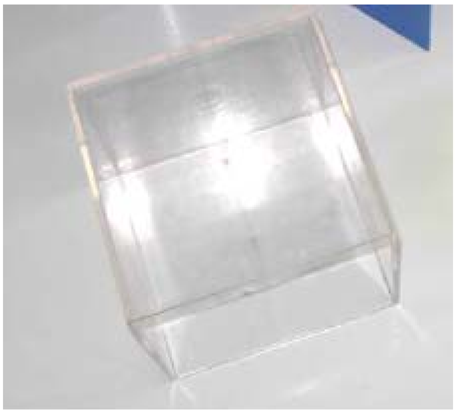
Thus, S (4.8/4.2) instead of (5.5/3.5). By the way, this value corresponds to the result of the example of task 5.

g) The analytical equation for the centre of gravity is valid for quadrangles with a regular mass distribution. $\vec{s} = \frac{1}{n\cdot m}\sum_{i=1}^{n} m\cdot\vec{x}_i = \frac{1}{n}\sum_{i=1}^{n}\vec{x}_i$

For example, if all four masses are equal, the result is $\vec{s} = \frac{1}{4}(\vec{x}_1+\vec{x}_2+\vec{x}_3+\vec{x}_4)$

Task 4: Centre of Gravity of Bodies

<u>Material:</u> Cube, cuboid and pyramid (if necessary other bodies), piece of string for hanging; if possible: computer with Mathematica programme for the construction of three-dimensional bodies and to check the position of the centre of gravity (animation possible).



<u>Task</u>: Determine the centre of gravity in the bodies at hand. First, make a hypothesis and then confirm it

a) by using the suspension method
b) physically, respectively, analytically
c) by using the computer

<u>Possible methods of resolution:</u>

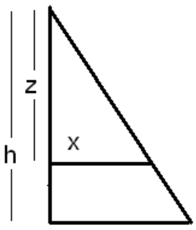
a) The hypothesis, according to which the centre of gravity is the point of interception of the symmetry axes, is confirmed if it is a cube. The centre of gravity of a pyramid is also on the symmetry axis. If a hollow and a solid body are used in the experiment, it could be found out that the centres of gravity may differ. For example, the centre of gravity of a solid pyramid of the height h is at ¼ h, while centre of gravity of a hollow pyramid depends on the size of base and height.
b) In solid bodies there is an even mass distribution, so that the physical centre of gravity corresponds to the analytical centre of gravity. However, the calculation is not easy, since we have a continuous mass distribution; an integral has to be used instead of a sum.

Suggestions for calculations:

<u>Solid Pyramid:</u>

Set the origin of the coordinate system at the peak of the pyramid, the z-axis pointing to the centre of the square base area.

Use the designations shown in the sketch one the right and intersection theorem:



$$\frac{x}{\frac{a}{2}} = \frac{z}{h} \Leftrightarrow x = \frac{a \cdot z}{2h} \Leftrightarrow y = \frac{a \cdot z}{2h}$$

Therefore the volume is:

$$V=\int_0^h\int_{-\frac{az}{2h}}^{\frac{az}{2h}}\int_{-\frac{az}{2h}}^{\frac{az}{2h}}dxdydz=\int_0^h\int_{-\frac{az}{2h}}^{\frac{az}{2h}}\frac{az}{h}dydz=\int_0^h\left(\frac{az}{h}\right)^2dz=\frac{1}{3}\frac{a^2}{h^2}h^3=\frac{1}{3}a^2h$$

The position of the centre of gravity $\vec{x}_S$ can be expressed by the volume in the following way (M = body's total mass, m_i = mass points with coordinates x_i, ρ= density)

Assuming the density is evenly distributed, it may be simplified to:

$$\vec{x}_S=\frac{1}{M}\sum m_i\vec{x}_i=\frac{1}{\rho V}\sum\rho V_i\vec{x}_i=\frac{1}{\int\rho dV}\int\rho\vec{x}dV \quad \text{thus:}$$

$$\vec{x}_S=\frac{1}{V}\int_0^h\int_{-\frac{az}{2h}}^{\frac{az}{2h}}\int_{-\frac{az}{2h}}^{\frac{az}{2h}}\begin{pmatrix}x\\y\\z\end{pmatrix}dxdydz=\frac{1}{V}\int_0^h\int_{-\frac{az}{2h}}^{\frac{az}{2h}}\begin{pmatrix}0\\\frac{az}{h}y\\\frac{az^2}{h}\end{pmatrix}dydz=\frac{1}{V}\int_0^h\begin{pmatrix}0\\0\\\frac{a^2z^3}{h^2}\end{pmatrix}dz=\frac{3}{a^2h}\begin{pmatrix}0\\0\\\frac{a^2}{h^2}\cdot\frac{1}{4}h^4\end{pmatrix}=\begin{pmatrix}0\\0\\\frac{3}{4}h\end{pmatrix}$$

So the centre of gravity of a solid pyramid is thus at height ¼ h.

<u>Hollow Pyramid</u>

Preliminary considerations: The origin of the coordinates is at the centre of the base area square ABCD, the z-axis pointing in the direction of the pyramid's peak E. Then coordinates of the corners are:

$$A\left(\frac{a\sqrt{2}}{2}/-\frac{a\sqrt{2}}{2}/0\right),\quad B\left(\frac{a\sqrt{2}}{2}/\frac{a\sqrt{2}}{2}/0\right),\quad C\left(-\frac{a\sqrt{2}}{2}/\frac{a\sqrt{2}}{2}/0\right),\quad D\left(-\frac{a\sqrt{2}}{2},-\frac{a\sqrt{2}}{2}/0\right),\quad E(0/0/0)$$

The pyramid consists of five areas. The area A of this surface area is the sum of the areas of the base square and the four side triangles, thus

$$A=a^2+4\cdot\frac{1}{2}a\cdot\sqrt{h^2+\frac{a^2}{4}}$$

Based on the calculations above, the position of the centre of gravity is determined by

$$\vec{x}_S=\frac{1}{M}\sum_{i=1}^{5}\vec{x}_i m_i=\frac{1}{\rho A}\sum_{i=1}^{5}\rho\vec{x}_i A_i$$

The centres of gravity of the individual areas result of the following calculations:

Square: $\vec{x}_{S(ABCD)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (point of interception of the symmetry axes)

Triangles, in general $\vec{x}_S = \frac{1}{3}(\vec{x}_1 + \vec{x}_2 + \vec{x}_3)$ and in this case:

$$\vec{x}_{S(ABE)} = \frac{1}{3}\begin{pmatrix} a\sqrt{2} \\ 0 \\ h \end{pmatrix},\ \vec{x}_{S(BCE)} = \frac{1}{3}\begin{pmatrix} 0 \\ a\sqrt{2} \\ h \end{pmatrix},\ \vec{x}_{S(CDE)} = \frac{1}{3}\begin{pmatrix} -a\sqrt{2} \\ 0 \\ h \end{pmatrix},\ \vec{x}_{S(DAE)} = \frac{1}{3}\begin{pmatrix} 0 \\ -a\sqrt{2} \\ h \end{pmatrix}$$

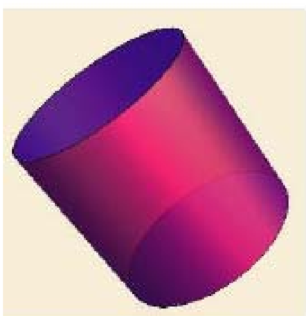
Centre of gravity calculation:
Since all of the four triangles have the same area, the following approach can be used:

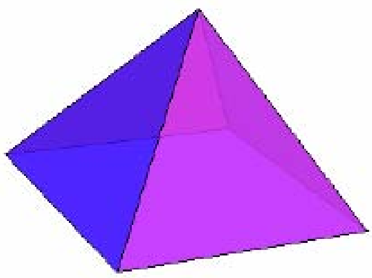
$$\vec{x}_s = \frac{1}{M}\sum_{i=1}^{4} m_i \cdot \vec{x}_i = \frac{4m_{triangle}}{M} \cdot \frac{1}{3}\begin{pmatrix} 0 \\ 0 \\ 4h \end{pmatrix} = \frac{4m_{triange}}{3M}\begin{pmatrix} 0 \\ 0 \\ 4h \end{pmatrix} = \frac{4\rho A_{triangle}}{3 \cdot (4\rho A_{triangle} + \rho A_{triangle})}\begin{pmatrix} 0 \\ 0 \\ 4h \end{pmatrix}$$

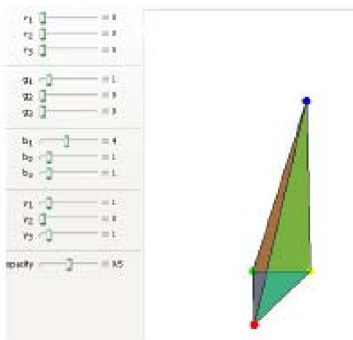
$$= \frac{4 \cdot 2a\sqrt{h^2 + \frac{a^2}{4}}}{3(4 \cdot 2a\sqrt{h^2 + \frac{a^2}{4}} + a^2)}\begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} = \frac{8h\sqrt{h^2 + \frac{a^2}{4}}}{3 \cdot (8\sqrt{h^2 + \frac{a^2}{4}} + a)}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus the coordinates of the centre of gravity depend on the measurements of the pyramid; but are close to 1/3 h. If, for example, a = h, the centre of gravity is $\frac{4}{3} \cdot \frac{\sqrt{5}}{1 + 4\sqrt{5}} h$, thus about 0.30h. If h = 2a the centre of gravity is at about 0.31 h. A slimmer pyramid with h = 8a, it is at 0.33 h etc.

c) If software, such as Mathematica, is available, the bodies can be generated in 3D-graphics, and the centre of gravity can be indicated directly via the menu. That way, calculations of the centre of gravity may be controlled and even the coordinates of different centres of gravity, caused by dynamic changes, can be investigated.







Worksheet – Centres of Gravity of Areas and Bodies

General Task
Determine the position of the centre of gravity of a triangle, quadrangle, cube, pyramid etc.

Task 1: Centre of Gravity of a Triangle

a) Cut a triangle out of a cardboard.
b) Draw the medians on the cardboard and then balance the triangle on your index finger in the medians' point of interception.
   Observe: What is the meaning of the medians' interception point?
c) Suspend the triangle from various points (corners or sides) and draw the respective lines of gravity, (if possible on the back, which hasn't been drawn on yet). Balance the triangle on your index finger at the interception point of the lines of gravity.
   Observe: What is the meaning of the interception point of the lines of gravity?
d) Confirm the identity of both points of interception.

Task 2: Analytical Description of the Triangle's Centre of Gravity

Any triangle ABC can be described via position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ starting at points A, B, C. Describe the position vector $\vec{s}$ of the centre of gravity S by $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Task 3: Centre of Gravity in a Quadrangle

Check, if the following (in analogy to the triangle) analytical description of the centre of gravity of a quadrangle is generally true: $\vec{s} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$

Carry out the following tasks:
a) Draw the trapeze ABCD with A (0/0), B (12/0), C (8/7) and D (2/7) on a cardboard.
b) Calculate the position of the centre of gravity using the equation given above.
c) Enter the centre of gravity in the cardboard trapeze on the basis of that result.
d) Check the position of the centre of gravity of the cardboard triangle experimentally by using the suspension method.
e) Check the position of the centre of gravity of the trapeze using the suspension method, too. Estimate the approximate coordinates. What do you notice?
f) The physical centre of gravity is given by the following equation:

$$\vec{s} = \frac{1}{m_1 + ... + m_k} \sum_{i=1}^{k} m_i \overrightarrow{x_i}$$

   m_i indicates the mass of body point i and x_i its positions.
   Explain the equation and use it to explain your observations in the experiments.
g) Answer the question above concerning the general validity of the equation for the centre of gravity of quadrangles

Task 4: Centre of Gravity in Bodies

Determine the centre of gravity of the bodies available. Express an assumption and check it afterwards by
a) using the suspension method
b) physically respectively analytically
c) using the computer.

Extension: Connection between Stability and Position of the Centre of Gravity
Stations: Phenomena

Hint: A lot of ideas are offered by special issues of school book publishers, but also literature for adolescents contains fascinating experiments concerning this theme.

Examples:

Shift of centre of gravity:

- Experiment 1: A balloon is filled with about 1 l of water and is then blown up. Two people then throw the balloon to each other. Through the movement of the water, the balloon's centre of gravity changes constantly, so that the direction of the balloon cannot be predicted.
- Experiment 2: A lemon is placed in a pot filled with water. Then, the test subjects tries to place a coin on top of the lemon in such a way that the coin does not drop into the water. By placing the coin on the lemon, the centre of gravity shifts, so that the lemon might turn over in worst case.

Stable balance: Centre of Gravity below the Point of Support

- Experiment 3: The test setting essentially consists of two parts:
  - Fixed part: A bottle is closed with a cork. A needle is carefully stuck in the cork, so that its tip points upwards.
  - Add-on part: A groove is carefully cut into the end of the cork with a knife and a coin is put into it. Then, two forks are symmetrically stuck into the cork on opposite sides.

  Execution of the test: The add-on part with the coin is placed onto the needle of the fixed part. The forks will be in balance, will even be able to rotate and won't fall down.

Stable balance: Centre of Gravity above the supporting area

- Experiment 4; A pile of books is placed on a table. The upper books are increasingly pushed forward, until the upper-most book no longer rests on the lowest book. One can push this pile of books to the edge of the table, without the pile collapsing. The stability results from the fact that the joint centre of gravity of the pile is above the supporting area.

This project has been funded with support from the European Commission. This publication reflects the views only of the author(s), and the Commission cannot be held responsible for any use which may be made of the information contained therein.