Education and Culture
Lifelong learning programme
COMENIUS

Teaching Material

Bicycle Gears as Ratios

About 30-40 years ago bicycles had no gears. Today bikes have many gears and it is much easier to ride a bike and achieve the optimal riding speed by appropriate gear shifting.

Today everybody has a sophisticated bike with gears. Modern bikes have up to 32 gears. A common number of gears is 21 and we shall consider such a bike in our presentation. From an early age we know how to shift gears in order to make a ride uphill easier and to go as fast as possible when we ride on a flat road or downhill. Basically, bike gear shifting means switching between three gears on the pedals axis and seven gears on the back wheel axis. This essentially means that with a gear ratio we make the back wheel go around from one to about four times, as we turn a pedal around once. One turn of a pedal for one turn of a wheel means a slow and strong motion, which is appropriate for an uphill ride. One turn of pedal for four turns of a wheel means a fast ride. Can we use mathematics to explain this?

Discussing these issues with a true bike at hand will make the experience and learning more vivid. Today we can use computers for smart simulations of real life situations.

A web version of the lesson with useful interactive simulations can be found at http://uc.fmf.uni-lj.si/com/BicGear/bicgear.html.

SIMPLE EQUATIONS

Let us start with a simplification. Assume we have a simple bike shifting system and only two gears at the front and two gears at the back. Say the front chainrings have 18 and 12 teeth and the back chainrings have 12 and 6 teeth. It also seems very natural to say that the rotation of the (front) pedals is an independent variable **x** and that the rotation of the back wheel, therefore the speed of the bike, is a dependent variable **y.**

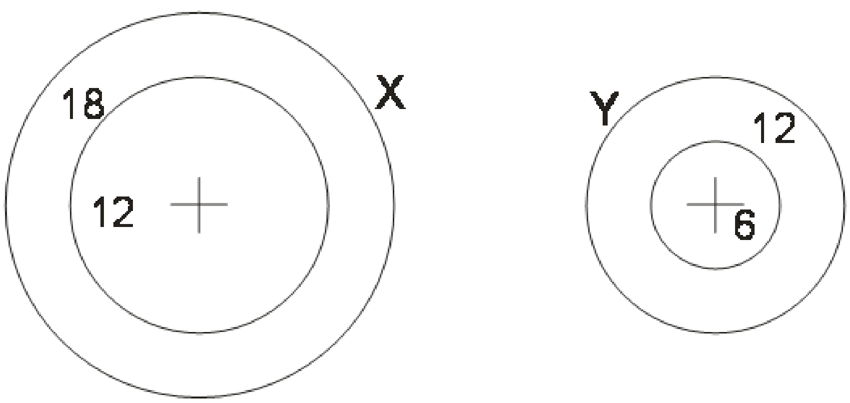

Figure 1: Front chainrings with 18 and 12 teeth and back chainrings with 12 and 6 teeth

In mathematics it is often that we solve equations like:

$$3\,x = 12$$

How do we do it? We divide the equation by *3* ... and we get that $x = 4$. There is not much understanding in it. But the above bicycle setting provides a possibility for deeper understanding of equations. Well, better, it provides a possibility for being aware of the deep meaning an equation can convey. If the chain connects the two 12-teeth chainrings, like on the picture bellow, it is easy to establish the equation.

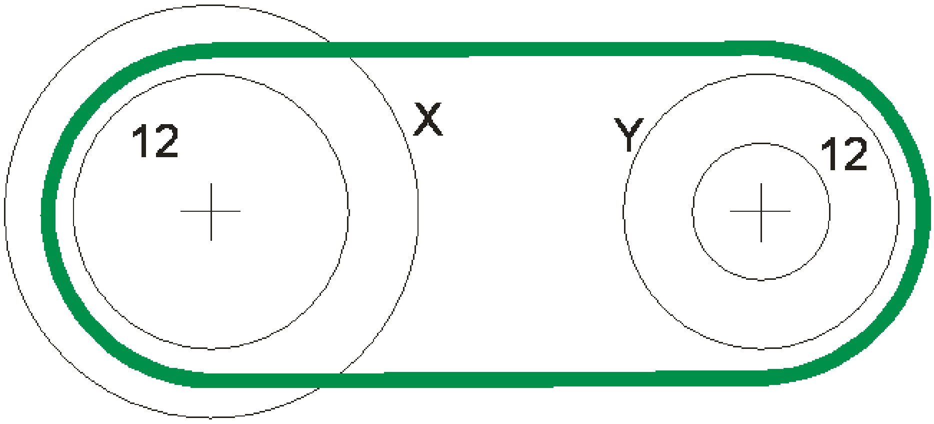

Figure 2: The chain connects the 12-teeth chainrings.

The equation would simply be

$$x = y$$

or

$$12\,x = 12\,y.$$

But do we know how to explain the setting with the two clogs not having the same number of teeth? Let us start with a case bellow.

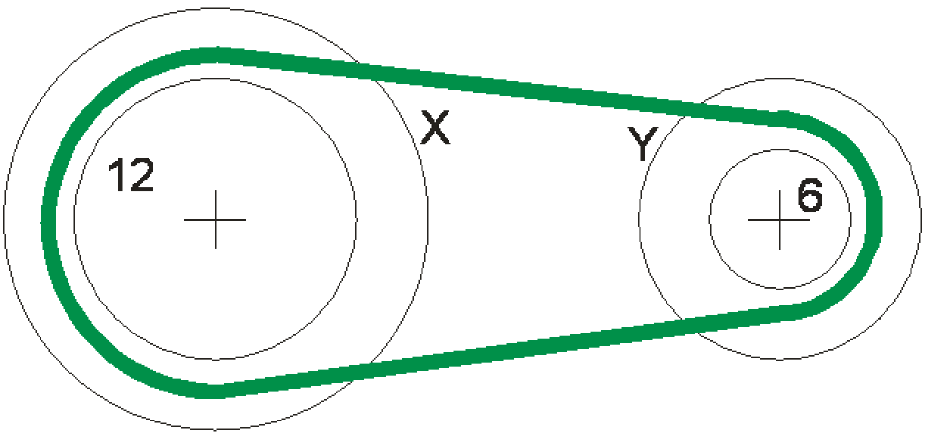

Figure 3: The chain connects the 12-teeth chainring in front with the 6-teeth chainring at the back.

What is the relation between **x** and **y** this time? Maybe it is only here that we became aware of the meaning of **x** and **y**. What do the two variables mean? It is only natural that **x** and **y** mean the 'number of rotations'. Therefore we have to count the teeth on the chainring, multiply the number of those with the number of rotations and we get the equation

$$12\,x = 6\,y$$

and

$$2\,x = y,$$

namely, the chain makes 'the number of moved teeth' equal. If this seems trivial, we advise teachers to test and ask the students to get the equation from the above picture without giving any advice. It is common that smart students might be able to give the right equation, by knowing what they should get rather than from understanding the relations given by 'chain connection of chainrings'.

THE RATIO OF BICYCLE GEARS

The best way to proceed would be to physically work on a real bike and count the teeth on all the chainrings. For the purpose of this presentation we counted the teeth on front and back clogs of a particular bicycle and we got the numbers, which we put into the following table. Here 1, 2, 3 and 1, 2, 3, 4, 5, 6, 7 mean the front and back gears and the numbers 30, 38, 50 and 34, 26, 24, 22, 20, 18, 15 mean the numbers of teeth on respective chainrings.

	front	1	2	3
back		30	38	50
1	34			
2	26			
3	24			
4	22			
5	20			
6	18			
7	15			

Figure 4: Horizontally we have front and vertically we have back chainrings.

As symbolically presented in figures 1-3, all the information about the bicycle gears can be presented by the ratios of front and back number of teeth on appropriate chainrings. These ratios tell us how many rotations of the wheel we get by one rotation of the pedals. Let us calculate the ratios and fill up the table.

	front	1	2	3
back		30	38	50
1	34	0.88	1.12	1.47
2	26	1.15	1.46	1.92
3	24	1.25	1.58	2.08
4	22	1.36	1.73	2.27
5	20	1.50	1.90	2.50
6	18	1.67	2.11	2.78
7	15	2.00	2.53	3.33

Figure 5: The ratios define gears.

The ratios in the table tell us how many rotations the wheel makes for one rotation of the pedals. Now we can measure the diameter of a wheel (in our case 70.5 cm) and let's assume we pedal with the speed of one rotation per second. An easy but 'real life' calculation tells us that one rotation of a wheel per second would give a travelling speed of

70.5 x 3.14 x 3600 cm/h = 7.97 km/h.

Multiplying the above ratios with this speed we obtain the chart of (rounded off) speeds in km/h:

	front	1	2	3
back		30	38	50
1	34	7.0	8.9	11.7
2	26	9.2	11.7	15.3
3	24	10.0	12.6	16.6
4	22	10.9	13.8	18.1
5	20	12.0	15.1	20.0
6	18	13.3	16.8	22.1
7	15	15.9	20.2	26.6

Figure 6: The speeds in km/h depending on a gear while pedalling with the speed of one rotation per second.

Comparing the numbers in either of these tables tell us the order of 21 gears on our bike. Let us take a careful look at the numbers in the ratio - table and fill in the consequential numbers from 1 to 21 in order as the gears follow.

	front	1	2	3
back		30	38	50
1	34	0.88 1	1.12 2	1.47 7
2	26	1.15 3	1.46 6	1.92 13

3	24	1.25	4	1.58	9	2.08	15
4	22	1.36	5	1.73	11	2.27	17
5	20	1.50	8	1.90	12	2.50	18
6	18	1.67	10	2.11	16	2.78	20
7	15	2.00	14	2.53	19	3.33	21

Figure 7: Twenty-one quotients consequentially

This is the place to discuss, weather we really have 21 gears on our bike. Theoretically yes, but are we able to explain the strange sequence of gears? How many times do we have to shift gears to shift from say from 7-th to 8-th gear? Is it six times? Yes! All together to follow the sequence of shifting consequentially from the 1-st to the 21-st gear, we would have to shift gears 58 times. That is surely not practical and very few people use all 21 gears. Let us explain the situation of bicycle gears by simple mathematics.

LINE SLOPES AND BICYCLE GEARS

Knowing that quotients mean actual speeds, let us draw the points in the regular coordinate system, where the numbers of teeth on the back chainrings are put on the x-axis and the number of teeth on the front chainrings are put on the y-axis. The steeper the slope, higher is the speed. As we are used to order and count the speeds from slower to faster, we should emphasize, that front speeds on our graph go from 1 to 3 in the standard upward direction, while the back speeds on our graph go from 1 to 7 in the direction from right to left.

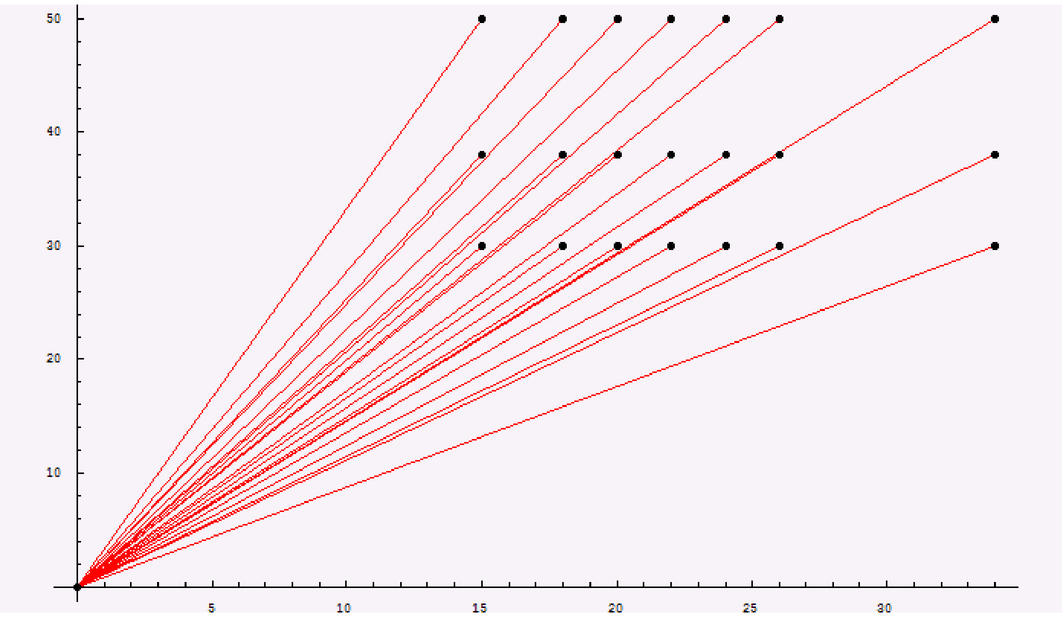

Figure 8: Quotients as slopes of lines

On the above graph it is easy to explain gears, shifting, and speeds. Different points mean different speeds. Shifting from one gear to another means moving from one point to another by either vertical or horizontal moves. Moving one unit to either vertical or horizontal direction means shifting one gear. Horizontal move means a back gear shift and vertical move means a front gear shift. On the graph it is easy to see, that some of the slopes are very close. Sure, different pairs of numbers can give equal or similar quotients. In our case all the quotients are different but the differences between some are very small. In reality this means that bicycle riding speed might be practically the same even if we have different combination of front and back gears. For example the 3-rd front and the 1-st back gear give basically the same ratio and basically the same speed as the 2-nd front and the 2-nd back gear.

Thus, our bicycle has many speeds, but some of them are basically the same. Therefore, as in real bicycle riding life, we only pick some of the quotients, which evenly cover the interval of ratios from the smallest to the biggest. On the picture bellow, we present those by green slope lines.

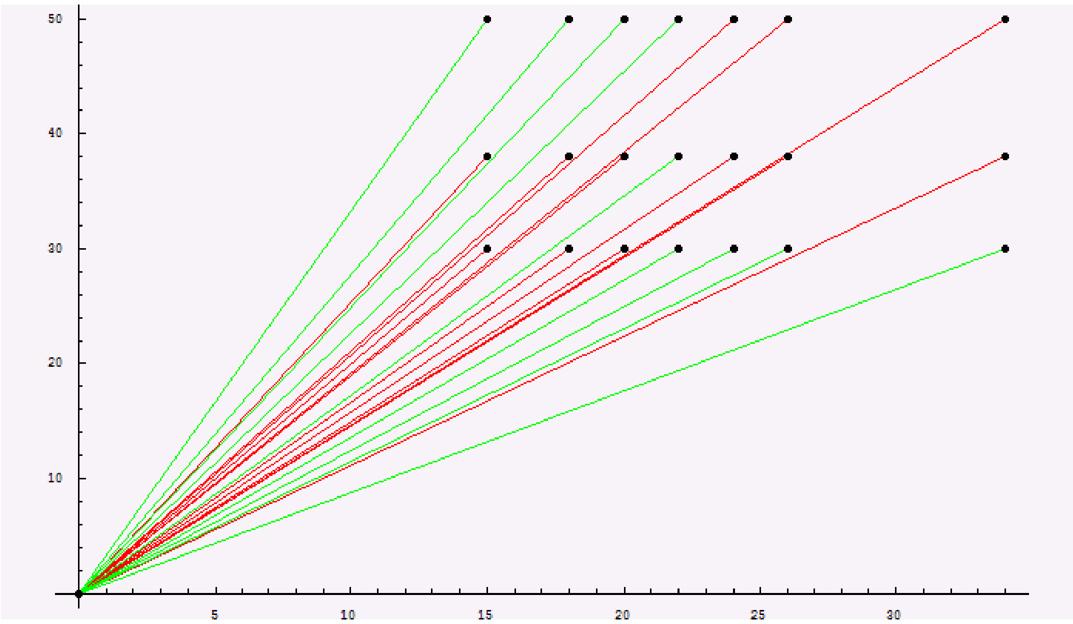

Figure 9: We do not really use all the gears

By skipping some of the gears we save lots of shifting. The gears presented are of course not the only 'workable' choice. At this level an interesting discussion can be had to deepen the understanding of the situation and especially of the presented graph. Teachers and students can be encouraged to present their 'gear shifting habits'. It is pretty straightforward to conclude that most sensible 'bicycle shifting habits' can be presented as a monotonous path from bottom-right to the upper-left corner.

Finally we present bellow our 'cleaned up' graph where we only have nine gears.

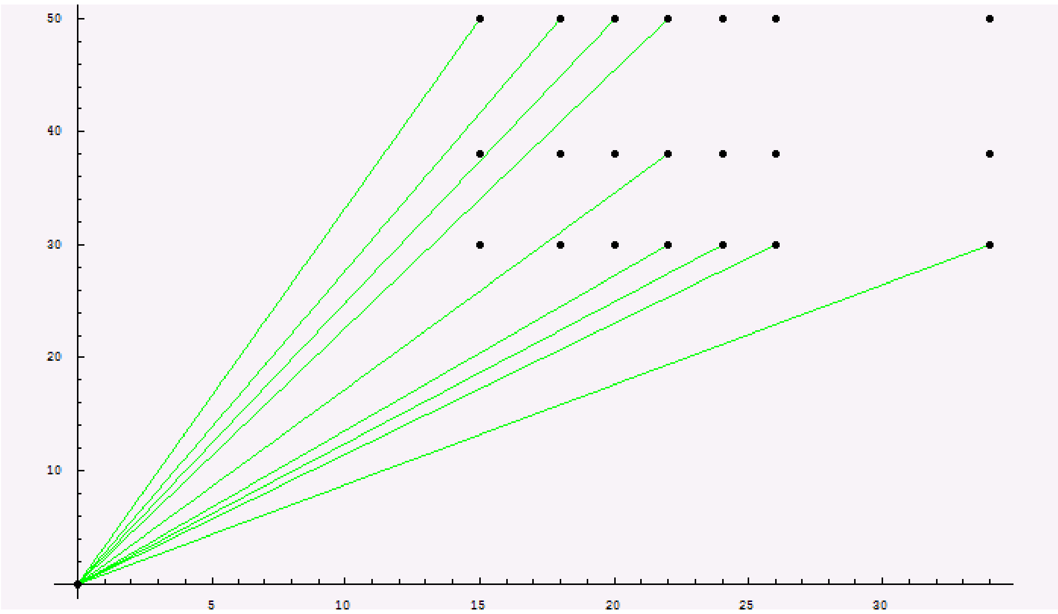

Figure 10: Nine gears and eight shifting

'Travelling' from upper-left to the bottom-right corner, our bicycle shifting could be described by 'right-right-right-down-down-right-right-right' sequence. It is pretty obvious, that for example 'right-right-down-right-right-down-right-right' would also be a sensible gear shifting. The discussion of different 'right-down' paths on our graph can be a good way of enhancing the understanding of coordinate system and the meanings graphs can convey.

Interactive simulation

Some further interesting interactive computer simulations of the ideas presented can be found and studied on the web.

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