The **ScienceMath-**project: **The Arithmetic Mean and Car Differential**
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Education and Culture
Lifelong learning programme
COMENIUS

Teaching Material

The Arithmetic Mean and Car Differential

The mechanism of a car differential (differential gear) has been known for about two thousand years. It represents an ingenious technical invention, which is nothing else than a realisation of a simple mathematical idea of the arithmetic mean. The study of a car differential provides a practical and intuitive insight into an otherwise abstract concept of variable dependency in simple mathematical equations.

A web version of the lesson can be found at http://uc.fmf.uni-lj.si/com/dif/cdif.html.

INTRODUCTION

All of us daily take advantage of the comfort, which is provided by technology and the car differential is an important technical device, which we all use regularly. But very few are aware of it or have ever contemplated this simple technical idea. And even fewer have ever thought about its natural connection to the very simple mathematical ideas. For anyone trying to understand mathematics, it can be of a great help if abstract mathematical ideas are given deeper meaning or its mechanical realisations that provide for our comfort living.

We all know that a car is powered by a motor; but how? How is the power (the rotation) of the motor transferred to the wheels that make the car move? On a bicycle, we use a chain that transfers the rotation of the pedal to the back wheel. Is it not done very similarly for a car, just that the source of power is a motor and not our muscles? Well, the very first cars were truly done that way. By the use of a chain the rotation was transferred from the motor to the (back) axle. So both, right and left wheel rotated simultaneously and made the vehicle to move.

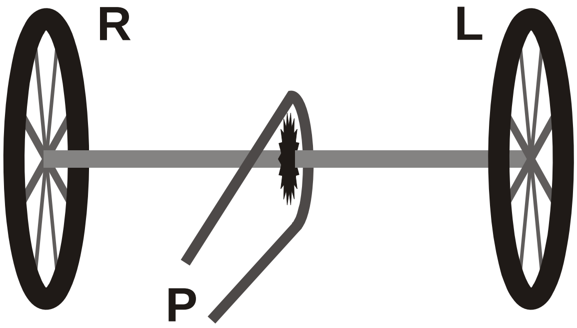

Figure 1: Right and left wheels are attached to the same axes of rotation

With a look at the picture, disregarding possible transmission ratio and denoting 'power' (engine rotation) by P, right wheel rotation by R and left wheel rotation by L, we get a very simple equation (system of equations):

$$P = R = L$$

But does a car work like that? Well, the very first cars did function like that and as a consequence, the steering was very hard. Namely, in a left turn, the right wheel travels longer route than the left and in a right turn, the situation is reversed. Theoretically, with a mechanism like on the above picture, the steering is impossible as both wheels rotate identically and thus, travel the path of the very same length. In reality, the steering is done while both wheels must slide slightly on the ground. That makes steering with such a mechanism physically quite hard. Today, one can experience this effect while driving a tractor or a jeep with a ‘blocked differential’. On a rough terrain, for example farmers must use their tractors with ‘blocked differential’ to increase the puling power. But while driving with ‘blocked differential’ it is very (physically) hard to move a steering wheel to either right or left turn position. In reality, several ‘tractor accidents’ are caused by weakness of the driver to turn the tractor while in a ‘blocked differential’ position. Sometimes a steering wheel and steering (usually front) wheels might even be turned in the right position, but the tractor with a ‘blocked differential’ might just push straight.

How is this problem solved in reality? A simple solution would be to transmit the engine rotation only to one of the two (right or left) wheels. That way steering would be ‘easy’ but for today’s standards of comfortable driving, driving and steering would be truly bizarre and dangerous. For example with the power on the right wheel, driving into a left turn would feel like really slowing down, while a turn to the right would be a stunt (a car would feel like speeding up into a right turn).

SOCIAL DIFERENTIAL

It might be an interesting moral discussion for a philosophy class, but thinking about our ‘build in’ social differential might turn out to be even technically and mathematically very intuitive and easy to understand. As with a car differential, we do not acknowledge that we have it if it works properly, but we become very aware of it when it brakes down.

We could say every normally developed human being has a ‘social differential’. Imagine a couple walking one next to another, chatting and not thinking about the path they walk possibly in a nice park, or even walking home using stairs all the way to the fourth floor and making sharp turns on each semi-floor. While turning left or right, both promenaders would (subconsciously) adopt their pace as to remain lined with a friend. If we think about what happens while turning, it is quite obvious that for example in a right turn, the right promenader would slow down a bit and the left would speed up a bit and their average speed would remain unchanged.

As mentioned above, malfunctioning of social differential can be noticed easily. Usually it happens with ‘very important’ people, who are too ‘important’ to adopt their speed to their subordinate and maintain their constant walking pace also when turning ... It is really funny to observe such a scène when for example ‘a subordinate’ student on the right side of ‘an important’ professor with a broken social differential runs in a left turn and remains almost still in a right turn.

THE ARITHMETIC MEAN

A (normal) promenading couple turns to the left so that the left promenader slows down and the right speeds up a bit. Their average speed remains unchanged. If P is their average speed, R is the speed of the right and L is the speed of the left promenader, then their ‘social differential’ is described by a simple equation:

$$P = \frac{R+L}{2}$$

The equation nicely and fully describes the relation between their speeds and average speed throughout the promenade, when their path is straight and their speeds are equal (in this case we have *P*=*R*=*L*) as well as when their speeds differ in left or right turns. Could this simple formula be mechanically realized for powering right and left wheels of a car? But how? It might be surprising, but a positive answer to this question has been known for over 2000 years. In fact, a mechanical realisation of the formula for arithmetic mean is surprisingly simple.

DIFFERENTIAL GEAR

Imagine first, that equal powering of the right and left wheels is achieved by a ‘rotating handle on a disc’, which is attached to the right and left disc, that are welded at the end of the right and left wheel axes, as shown on the bellow picture.

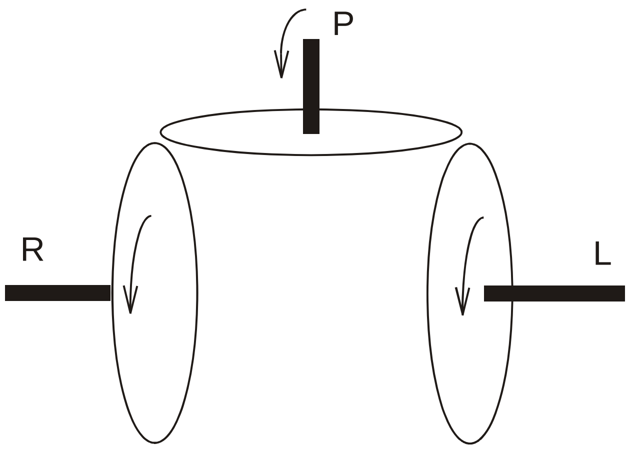

Figure 2: Right, left and the powering discs

Instead of discs we can imagine cogs. It is obvious, that with a help of such a mechanism, a rotation of ‘the handle P’ would imply an equal rotation of the left and right wheel, thus *P*=*R*=*L*. This seems like still far away from the desired equation:

$$P = \frac{R+L}{2}$$

But it is not. Starting with this formula

$$P - R = L - P = \frac{L - R}{2}.$$

we can easily check, that denoting

$$\frac{L - R}{2} = X,$$

we have

$$R = P - X \text{ and } L = P + X.$$

The value of *X* can be understood as a free parameter in the relation of three variables within a single equation

$$P = \frac{R + L}{2}.$$

The variable *X* has such an important role in the mechanic realisation of the arithmetic mean, that its understanding completely resolves the dilemma of the powering of the car wheels.

Namely, if we allow that our 'power disc' in the above picture, is freely rotatable (free variable *X*) around the 'handle', as shown in the picture below, we already have a model of a differential gear.

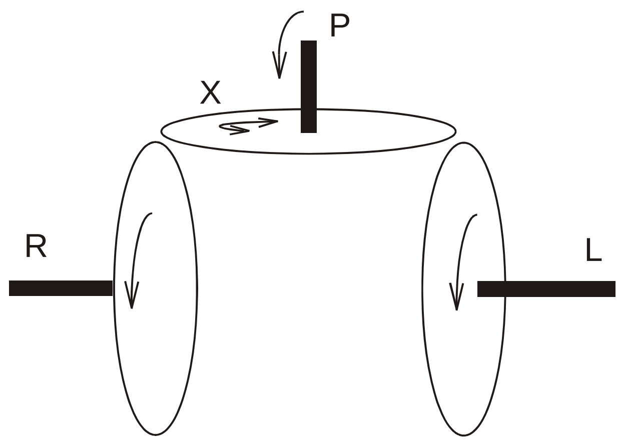

Figure 3: Right, left and freely rotatable powering discs

With a look on the picture above, let us think again about the formula

$$P = \frac{R + L}{2}.$$

If both right and left discs are freely rotatable, the push (rotation) of our handle will cause both discs to rotate evenly and the powering disc will not rotate (*X=0*). If either right or left discs is stopped (or only partially braked), the powering disc will start rotating as we push (rotate) the handle and the opposite disc will rotate even faster. Thus, 'variable *X*' will exactly transform for example slowing down of the right disc into speeding up of the left disc.

The above simple sketch illustrates the essence of a differential gear and gives a mechanic realisation of the simple but abstract *arithmetic mean* mathematical idea. Freely revolvable powering disc takes care of *differentiating* the resistance on the left and right half shafts. As much as one of the wheel torques (left or right) is diminished because of the resistance, as much the other is increased. The question of how to transmit the engine torque through the (cardan) drive shaft to our powering disc is not trivial, but regarding the described ingenious idea, this question is only technical.

On the bellow picture one can see a sketch of a true (classical) differential gear.

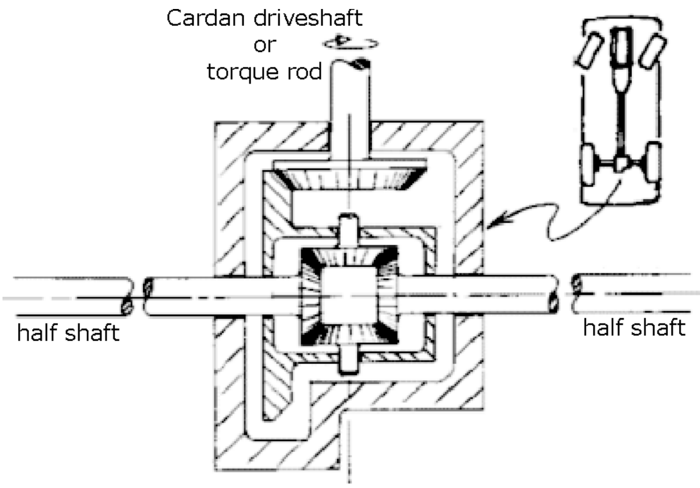

Figure 4: Differential gear profile

Engine torque is transmitted over the *cardan driveshaft* named also *torque rod*. It is interesting that the name *cardan* is directly derived from the name of Italian mathematician, physician and inventor *Girolamo Cardano* (1501-1576), who invented the *universal joint*. *Universal joint* is an essential part of a usable *torque rod*, but one that is not associated directly to our idea of a differential gear. *Universal joint* is another ingenious idea which provides a simple solution for 'around the corner rotation'. In other words it is a joint joining two simultaneously rotatable rods that are joined under an angle between 90 and 180 degrees.

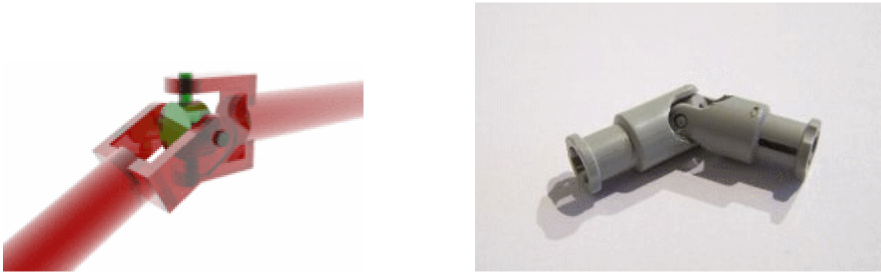

Figure 5: Cardan – a universal joint, drawing and Lego swivel

We believe the idea of a differential gear can be a useful didactical motivational tool. It provides a useful, complex and yet simple technical and intuitive idea from where we can derive and contemplate deeper meanings of otherwise only abstract mathematical ideas. Namely, even to a mathematics expert, this simple idea of arithmetic mean can pose several quite nontrivial questions. Furthermore, abstract ideas can be given intuitive and technical meanings via well known questions related to common experience of driving, turning. One can easily experiment as today, even Lego (Technics) provides sophisticated but yet simple models of devices like differential gear.

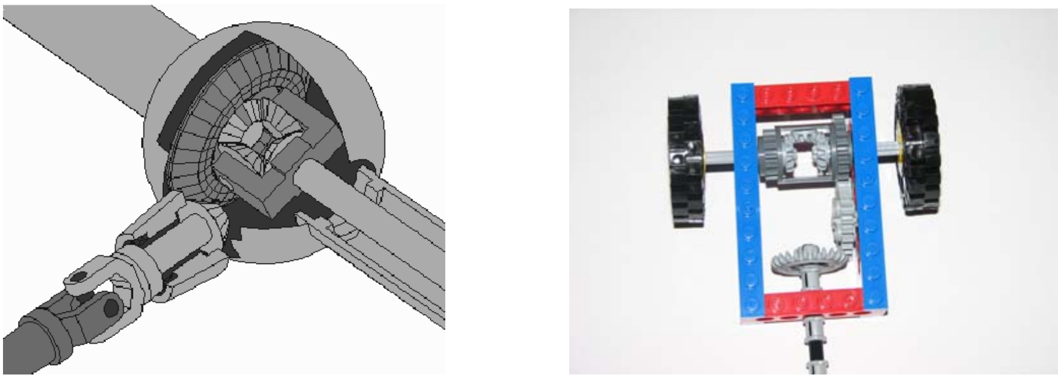

Figure 6: Drawing of a differential gear with universal joint and Lego model of differential

ARITHMETIC MEAN AND SNOW DRIVING

Many people have experiences related to the functioning of a differential gear mechanism. That might be tractor or jeep driving as mentioned at the beginning of this article, but more often and unfortunately unpleasant are experiences of a car driving in a snow. It happens very easily that driving in a snow leaves us powerless on the road, when we are unable to move the car. The engine would just helplessly rotate one of the wheels, which would freely slide on the smooth snow. Usually this happens when a car leans to one side and whatever we tray, the car just sinks deeper into the snow. Always it is only one wheel that rotates, and even that is the wrong one. If the wheel on the side where the car leans would rotate, the force might be strong enough to move the car forward... It even happens that we get some strong help and powerful boys try to lift the side of the car that is burdened because of the lean. At the same time they might push down the other side of the car to put pressure on the spinning wheel... Sometimes it might help but even more common is the situation that shifting the leaning of the car to the other side only causes the shift from one freely spinning wheel to the other. Well, this is the situation when a differential gear is doing right the opposite of what would be productive. Namely, differential gear always makes the easier rotatable wheel to spin. Of course all this wrestling can be easily explained by the arithmetic mean formula

$$P = \frac{R + L}{2}.$$

These thoughts can be an insightful start of other mathematical chapters, like for example system of equations. Namely, considering P as given (constant – engine power), this is really a good and simple example, which tells us that one equation can only tell us the relation but not the absolute values of the two variables it connects. Thus a natural need for two equations to determine two variables is given.

There are further interesting questions that one can consider with students. For example:

A car with a turned off engine, no hand brakes applied in forward gear position: Can it be pushed forward? By experience, many would answer correctly, that the car can not be pushed..., but few would understand, how this could be implied from the equation

$$P = \frac{R+L}{2}.$$

The next question gives simple interpretation of that.

Let a car be in exactly the same position as above, but rather than on the ground, let us imagine the car is lifted up as in a garage. Can a powering wheel be turned around by hand?

It is very interesting that usually only 'technically inclined professionals' answer correctly to this question. Even mathematics teachers after workshops of work on this idea, are deceived by misunderstanding of their experience. Namely, in the above described position, powering wheel can easily be turned around..., while the opposite wheel turns to the opposite direction. Of course, since the car engine is off and the car is in forward gear position *P* in our equation

$$P = \frac{R+L}{2}$$

is forced to be 0. Thus $R = -L$. So how come that forward (or backward) gear position works as a break of a car standing on a road? Well, of course, moving a car forward or backward would mean that both left and right wheel turn to the same direction, while when *P=0*, *R* and *L* can only be of the same sign if they are both 0.

DIFFERENTIAL, THE STRAIGHTNESS CONCEPT AND GEODESICS

It is interesting that such a simple idea can be developed further into a wonderful intuitive understanding of a complex and abstract differential geometry concept of geodesics. A chapter that is a hard task for university students. Before that high school students can be engaged into a debate over what *straight* means in reality. In mathematics we know the idea of a straight line, but in real life, do we know anything that is 'more straight' than the 'equator circle'. From here it is easy to derive a concept of '*straightness*' as the shortest distance... Of course, on a plane that is a straight line. On a sphere *straightness* is best described by great circles. This also explains why long distance airplanes fly 'strange arced paths' on our usual maps. It took a long time to mankind to comprehend that a straight edge of a table is no straighter to a man than a highway loop around a small town is to an ant. Formal definition of a straight line is quite complicated and abstract. On a surface, which is by our experience 'flat' but by our limited understanding in fact its curvature remains unknown to us; we define straightness as *the shortest distance*. The shortest paths are called geodesics. And what has this to do with our differential gear? As described in the very beginning of this article, a tractor or a jeep with a blocked differential would only drive straight. As would a simple Lego model with two wheels attached to the same axes, if carefully pushed, only go straight. But what if the 'driving ground' is not flat?

Well, then our vehicle would travel wonderful intuitive paths of geodesics... posing many further questions, inspiring our imagination and challenging our understanding.

HISTORIC REMARKS

It is not known who invented differential gear mechanism. It seems obvious that the idea is much older than many of Leonardo da Vinci's (1452-1519) inventions. British inventor James Starley (1830-1881), known as 'Father of the Bicycle Industry', used a differential gear mechanism in a special sewing machine in about 1850. In 1877 he used the differential gear in a road vehicle. Supposedly, differential gear was used in a road vehicle for the first time by German Rudolph Ackerman in 1810. Several sophisticated mechanical devices that included differential gear mechanism are much, much older. Findings in China prove the existence of this mechanism dating back to about 300. In the year 1900 an extremely sophisticated **Antikythera mechanism** (named after nearby Antikythera island of Greece, where the ship was discovered) was found in a ship wreck. The mechanism was a carefully designed and crafted in bronze and wood. It was a sort of astrological computer to calculate the position of planets and stars. And what is the most amazing, a device has been dated to about 125 BC and it had a differential gear mechanism. The device is displayed in the Bronze Collection of the National Archaeological Museum of Athens, while several exact reconstructions have been made and are on display around the globe, usually in computer museums (like for example in American Computer Museum in Bozeman, Montana).

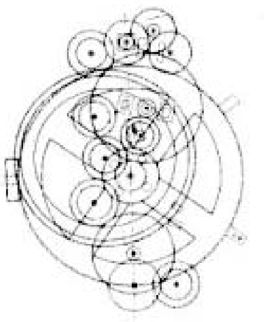

Figure 7: The main fragment of The Antikythera mechanism (~125 BC) and its reconstructed plan

TEACHING MATERIAL

It is a wonderful motivation and a way of practical understanding of the arithmetic mean if students are introduced with 'hands on' Lego models. With those, many above questions and comments will gain intuitive and abstract understanding. It is quite demanding to get enough adequate Lego models for students to work practically, but it is worth the effort. It is best if one or two students have a model to build and analyse. And we should start slow and didactically challenging by first building 'chariots' with two wheels on the same and on separate axes, as mentioned above. Experimenting with such 'chariots' and possibly smartly performed 'walks of two' observing the different speeds, or 'broken social differentials' might bring fun into an entertaining but valuable lesson. Above mentioned Lego differentials are included in some of the commercial Technics models and can be acquired via special school Lego sets.

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Interactive simulation

Interactive computer simulation of a car differential can also be reached and studied on the web.

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